Chapter 5. Stratified Quasi-Geostrophic Rossby Waves

Sec. 5.1: Quasi-Geostrophic Equation in Stratified Fluid

1. Nondimensional Equations

We'll use the oceanic equations (4.1.15)

$$\begin{aligned}
&\partial_t u + (\mathbf{u}\bullet\nabla)u - fv = -\frac{1}{\rho_m}\partial_x p_0 + \frac{1}{\rho_m}F_x \\
&\partial_t v + (\mathbf{u}\bullet\nabla)v + fu = -\frac{1}{\rho_m}\partial_y p_0 + \frac{1}{\rho_m}F_y \\
&\frac{1}{\rho_m}\partial_z p_0 = -\frac{g\rho_0}{\rho_m} \\
&\partial_x u + \partial_y v + \partial_z w = 0 \\
&\partial_t \rho_0 + (\bar{\mathbf{u}}\bullet\nabla)\rho_0 = S_0
\end{aligned} \tag{5.1.1}$$

We choose the scales as

$$u,\ v \sim U, \quad x,y \sim L, \quad w \sim U\frac{D}{L}, \quad z \sim D, \quad t \sim \frac{L}{U}$$

and denote the Rossby number as:

$$\varepsilon = \frac{U}{f_o L}$$

The density and pressure can be written as

$$\rho_0(x,y,z,t)\ (= \rho_{Ocean} - \rho_m) = \rho_S(z) + \rho(x,y,z,t), \tag{5.1.2a}$$

$$p_0(x,y,z,t)\ (= P_{Ocean} + \rho_m g z) = p_S(z) + p(x,y,z,t), \tag{5.1.2b}$$

where ρ_{Ocean} and P_{Ocean} are the total density and pressure, ρ_m is the average density of the ocean, and

$$\frac{dp_S}{dz} = -g\rho_S(z)$$

represents the static part associated with the mean stratification and

$$\frac{dp}{dz} = -g\rho$$

is the dynamic pressure associated with the horizontal density variations. For large scale flows with small Rossby number, similar to the shallow water case in section 2.1, the dynamic pressure is also scaled as

$$p \sim f_o L U \rho_m \tag{5.1.3}$$

such that the pressure gradient force is comparable with the Coriolis force; the source/sink is assumed weak, with $\frac{1}{f_o U \rho_m}\mathbf{F} = \varepsilon \mathbf{G}$, and $G \leq O(1)$; we also use the local β-plane approximation $\frac{\beta L}{f_o} = \gamma\varepsilon$, where $r \sim 1$. Then, we can write the two momentum equations in dimensionless variables (subscripted with "*") as:

$$\varepsilon\{\frac{\partial u_*}{\partial t_*} + (\mathbf{u}_* \bullet \nabla_*)u_* - \gamma y_* v_*\} - v_* = -\frac{\partial p_*}{\partial x_*} + \varepsilon G_x$$

$$\varepsilon\{\frac{\partial v_*}{\partial t_*} + (\mathbf{u}_* \bullet \nabla_*)v_* + \gamma y_* u_*\} + u_* = -\frac{\partial p_*}{\partial y_*} + \varepsilon G_y$$

The continuity equation is

$$\frac{\partial u_*}{\partial x_*} + \frac{\partial v_*}{\partial y_*} + \frac{\partial w_*}{\partial z_*} = 0$$

The scale of ρ can be derived from the hydrostatic equation

$$\frac{\partial p}{\partial z} = -g\rho.$$

Notice (5.1.3), we have

$$\rho \sim \frac{p}{gD} \sim \frac{f_o L U}{gD}\rho_m \sim \rho_m (\frac{L}{L_D})^2 \varepsilon \equiv \Gamma \tag{5.1.4}$$

where $L_D{}^2 = \frac{gD}{f_o^2}$ is the *external* deformation radius. We can therefore define ρ as

$$\rho \sim \Gamma \rho_* \tag{5.1.4a}$$

where $\rho_* \sim O(1)$. The hydrostatic equation can be written as

$$\frac{\partial p_*}{\partial z_*} = -\rho_*$$

With all these scalings, the thermodynamic equation becomes

$$\frac{U\Gamma}{L}\{\frac{\partial\rho_*}{\partial t_*}+(\mathbf{u}_*\bullet\nabla_*)\rho_*\}+\frac{UD}{L}w_*\frac{d\rho_S}{dz}=S_0.$$

Thus,

$$\varepsilon\{\frac{\partial\rho_*}{\partial t_*}+(\mathbf{u}_*\bullet\nabla_*)\rho_*\}+\frac{UD}{f_oL\Gamma}w_*\frac{d\rho_S}{dz}=\frac{S_o}{f_o\Gamma}$$

$$\varepsilon\{\frac{\partial\rho_*}{\partial t_*}+(\mathbf{u}_*\bullet\nabla_*)\rho_*\}+\frac{D\varepsilon}{\Gamma}w_*\frac{d\rho_S}{dz}=\frac{S_o}{f_o\Gamma} \quad (5.1.5)$$

The scale of $\frac{d\rho_S}{dz}$ can be derived at the first order from the adiabatic condition. We can show from (5.1.5) that

$$\frac{d\rho_S}{dz}\geq\frac{1}{\varepsilon}\frac{\partial\rho}{\partial z}>>\frac{\partial\rho}{\partial z}$$

Indeed, for adiabatic flows, the solution is always along isopycnals $\frac{d\rho_0}{dt}=0$. Therefore, $u\partial_x\rho_0\approx w\partial_z\rho_0$. But, we know that QG equations require at the first order non-divergent:

$$\frac{w}{u}\leq\varepsilon\frac{D}{L}$$

Therefore, the slope of the isopycnal surface must be $\leq\varepsilon\frac{D}{L}$

$$\frac{\partial\rho_0}{\partial x}\Big/\frac{\partial\rho_0}{\partial z}\approx\frac{w}{u}\leq\varepsilon\frac{D}{L}$$

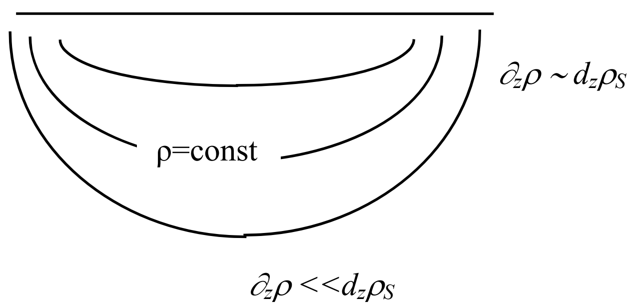
Since $\frac{\partial\rho_S}{\partial x}=0$, we have $\frac{\partial\rho_0}{\partial x}=\frac{\partial(\rho_S+\rho)}{\partial x}=\frac{\partial\rho}{\partial x}$. Furthermore, $\frac{\partial\rho}{\partial x}\approx\frac{\partial\rho}{\partial z}\frac{D}{L}$, we have

$$\frac{\partial\rho}{\partial z}\Big/\frac{\partial(\rho_S+\rho)}{\partial z}\leq\varepsilon$$

Since $\frac{\partial\rho}{\partial z}<<\frac{d\rho_S}{\partial z}$, we have

$$\frac{\partial\rho}{\partial z}\Big/\frac{d\rho_S}{dz}\leq\varepsilon \quad (5.1.6)$$

This implies that any horizontal variation of the static stability (since ρ is the horizontal variation part) must be small. This is a weak assumption in many cases. (especially in marine eddies ..)



Using (5.1.4) and (5.1.6), we have the scale

$$\frac{d\rho_S}{dz} = \frac{\Gamma}{D\varepsilon}\Gamma_*(z)$$

where $\Gamma_*(z) \le O(1)$.

The thermodynamic equation becomes

$$\varepsilon\{\frac{\partial \rho_*}{\partial t_*} + (\mathbf{u}_* \nabla_*)\rho_*\} + w\Gamma_* = \varepsilon S$$

where $\varepsilon S = \dfrac{S_0}{f_o \Gamma}$ and $S \le O(1)$ such that at the leading order the flow is adiabatic.

Note 1:The buoyancy frequency *N(z)* (Brunt-Vaisala frequency) is defined as

$$N^2 = -\frac{g}{\rho_m}\frac{d\rho_S}{dz}$$

Note: For a typical atmospheric tropospheric and upper oceanic stratification, we have

$$\frac{\delta\rho_{S,atm}}{\rho_m} \sim 0.1,\, D_{atm} \sim 10km, \quad \frac{\delta\rho_{S,ocn}}{\rho_m} \sim 0.001, D_{atm} \sim 1km$$

So, the BV frequency is therefore

$$N_{atm} = \sqrt{0.1 \times g/10^4 m} \sim 1/(300s) \sim 1/(5\min), \quad N_{ocn} = \sqrt{0.001 \times g/10^3 m} \sim 1/(1000s) \sim 1/(15\min)$$

||

$$O(1) \ge \Gamma_*(z) = \frac{\dfrac{d\rho_S}{dz}}{\Gamma / D\varepsilon} = \frac{N^2 \rho_m / g}{\left(\rho_m L^2 \varepsilon f_0^2 / gD\right) / D\varepsilon} = \left(\frac{L_I}{L}\right)^2$$

we have

where $L_I^2=(ND/f)^2$ is the *interior* deformation radius. Therefore,

$$\frac{\partial\rho}{\partial z} \le \varepsilon \frac{d\rho_S}{dz} \quad \Leftrightarrow \quad \left(\frac{L}{L_I}\right)^2 \le 1$$

This requires the scale to be not too large, similar to the homogeneous case.

The complete set of dimensionless equations are therefore (drop the subscript "*"):

$$\begin{aligned}
&v - \frac{\partial p}{\partial x} = \varepsilon\{\frac{\partial u}{\partial t} + (\mathbf{u}\bullet\nabla)u - ryv - G_x\} \\
&-u - \frac{\partial p}{\partial y} = \varepsilon\{\frac{\partial v}{\partial t} + (\mathbf{u}\bullet\nabla)v + ryu - G_y\} \\
&-w\Gamma_* = \varepsilon\{\frac{\partial\rho}{\partial t} + (\mathbf{u}\bullet\nabla)\rho - S\} \qquad (5.1.7) \\
&\frac{\partial p}{\partial z} = -\rho \\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\end{aligned}$$

Similar to the shallow water case in Section 2.1, we solve this set of equations by expanding variables as powers of ε:

$$\begin{aligned}
&u = u_o + \varepsilon u_1 + O(\varepsilon^2) \\
&v = v_o + \varepsilon v_1 + O(\varepsilon^2) \\
&p = p_o + \varepsilon p_1 + O(\varepsilon^2) \ . \qquad (5.1.8) \\
&\rho = \rho_o + \varepsilon\rho_1 + O(\varepsilon^2) \\
&w = w_o + \varepsilon w_1 + O(\varepsilon^2)
\end{aligned}$$

2. O(1) Equation and Dynamics

Substitute (5.1.8) into (5.1.7), at the leading order, we have

$$
\begin{aligned}
v_o &= \frac{\partial p_o}{\partial x} \\
u_o &= -\frac{\partial p_o}{\partial y} \\
w_o &= 0 \\
\frac{\partial p_o}{\partial z} &= -\rho_o \\
\frac{\partial u_o}{\partial x} + \frac{\partial v_o}{\partial y} + \frac{\partial w_o}{\partial z} &= 0
\end{aligned}
\qquad (5.1.9\text{a-e})
$$

As in the shallow water case, (a), (b), (c) can be used to derive (e). Thus, there are only 4 independent equations, but with 5 unknowns. This is the “Geostrophy degeneracy”.

To better understand the O(1) dynamics, we write (5.1.9a,b) in dimensional form as the geostrophic balance:

$$
\begin{aligned}
v_g &= \frac{1}{f_o \rho_m} \frac{\partial p}{\partial x} \equiv \partial_x \psi \\
u_g &= -\frac{1}{f_o \rho_m} \frac{\partial p}{\partial y} \equiv -\partial_y \psi
\end{aligned}
\qquad (5.1.10)
$$

where $\psi = \frac{p}{\rho_m f_o}$ is the geostrophic stream function. The hydrostatic balance (5.1.9c) can be written as

$$
\frac{\rho}{\rho_m} = -\frac{f_o}{g} \frac{\partial \psi}{\partial z} \qquad (5.1.11)
$$

Differentiate (5.1.10) with respect to *z* and use (5.1.11), we have

$$
\begin{aligned}
\frac{\partial v_g}{\partial z} &= \frac{\partial^2 \psi}{\partial x \partial z} = -\frac{g}{f_o \rho_m} \frac{\partial \rho}{\partial x} \\
\frac{\partial u_g}{\partial z} &= -\frac{\partial^2 \psi}{\partial y \partial z} = \frac{g}{f_o \rho_m} \frac{\partial \rho}{\partial y}
\end{aligned}
\qquad (5.1.12)
$$

This is the thermal wind relation, a direct result of geostrophy and hydrostatic balance. This relation has been used frequently to infer ocean currents from the density field, i.e. the so called “dynamic method”.

Note 2: It also represents a balance between the baroclinic vorticity generation and the title of planetary vorticity in y- and x- directions. Indeed, the general vorticity equation is:

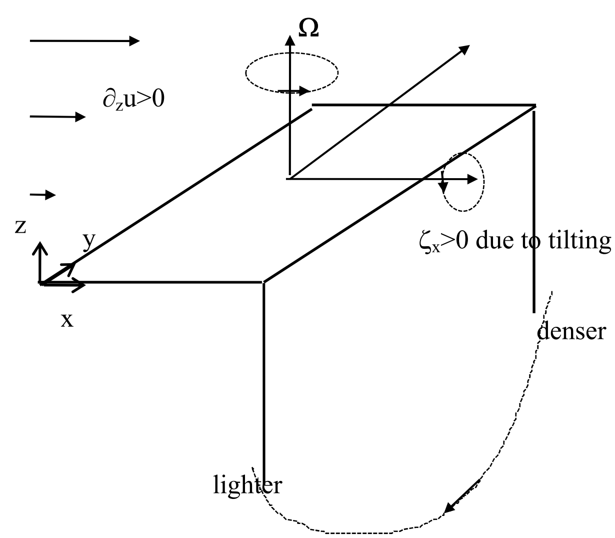
$$\frac{d\boldsymbol{\xi}_a}{dt} = (\boldsymbol{\xi}_a \bullet \nabla)\mathbf{u} - \boldsymbol{\xi}_a \nabla \bullet \mathbf{u} + \frac{\nabla\rho \times \nabla p}{\rho^2}$$

For large scale $\varepsilon << 1$, we have $\boldsymbol{\xi}_a = 2\boldsymbol{\Omega} + O(\varepsilon)$. In addition, $\partial_z\rho\partial_y P << \partial_y\rho\partial_z P$, which is equivalent to the Boussinesq approximation in the ocean. The x and y component of the vorticity equation can therefore be written as

$$\frac{d\xi_{ax}}{dt} \approx 2\Omega\partial_z u + \frac{\partial_y\rho\partial_z p - \partial_z\rho\partial_y p}{\rho^2} \approx 2\Omega\partial_z u + \frac{\partial_y\rho\partial_z p}{\rho^2}$$

$$\frac{d\xi_{ay}}{dt} \approx 2\Omega\partial_z v + \frac{\partial_z\rho\partial_x p - \partial_x\rho\partial_z p}{\rho^2} \approx 2\Omega\partial_z v - \frac{\partial_x\rho\partial_z p}{\rho^2}$$

Notice hydrostatic balance: $\partial_z p = -\rho g$, in the steady state $\frac{d\xi_{ax}}{dt} = \frac{d\xi_{ay}}{dt} = 0$, we have the thermal wind relationship. On the RHS in the two equations above, the first term is the tilting term while the second term is the baroclinic term. The balance can be seen schematically as follows:



A westerly wind shear $\partial_z u > 0$ generates positive vorticity in the x direction $\Omega\partial_z u > 0 \Rightarrow \partial_x \xi > 0$. This vorticity is balanced by the opposite rotation that is forced by the northward density gradient $\partial_y \rho > 0, \partial_z p < 0 \Rightarrow \partial_y \rho \partial_z p < 0$.

Note 3: Taylor - Proudman Theorem

For large scale low frequency processes, we have $\xi_a \approx 2\mathbf{\Omega}$, and $\partial_t < \Omega$. The vorticity equation, assuming incompressibility, is

$$(2\mathbf{\Omega} \bullet \nabla)\mathbf{u} = \frac{\nabla\rho \times \nabla p}{\rho^2}$$

If furthermore, the fluid is barotropic $\nabla\rho \times \nabla p = 0$, we have

$$(\mathbf{\Omega} \bullet \nabla)\mathbf{u} = 0$$

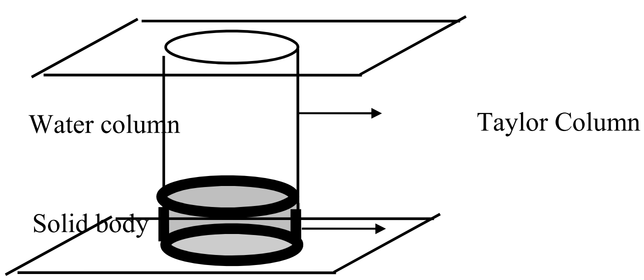
or assuming $\mathbf{\Omega} = \Omega\mathbf{k}$, this is

$$\Omega\partial_z \mathbf{u} = 0$$

or there is no shear of velocity in the vertical direction.

$$\partial_z u = \partial_z v = \partial_z w = 0$$

The water column therefore behaves like a column of solid body.



Even in stratified case, this still shows the tendency of rotation to constraint fluid motion variation along **Ω.** In other words, rotation tends to couple the flow in the direction of **Ω**. Thus, rotation produces a "stiffening" effect that tends to align the vortex tube in the direction of rotation. This is also why in a layered model we can assume no shear for large scales GFD processes.

<u>3. O(ε) Equation and QGPV Equation</u>

At the next order, we have the equations

$$v_1 - \frac{\partial p_1}{\partial x} = \frac{D_{*g}}{Dt} u_o - ryv_o - G_x$$

$$-u_1 - \frac{\partial p_1}{\partial y} = \frac{D_{*g}}{Dt} v_o + ryu_o - G_y$$

$$-w_1 \Gamma_* = \frac{D_{*g}}{Dt} \rho_o - S \qquad (5.1.13)$$

$$\frac{\partial p_1}{\partial z} + \rho_1 = 0$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0$$

where $\frac{D_{*g}}{Dt} = \frac{\partial}{\partial t} + u_o \frac{\partial}{\partial x} + v_o \frac{\partial}{\partial y}$. (This becomes a horizontal total derivative because $w_o = 0$). The vorticity equation is therefore:

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = -\frac{D_{*g}}{dt}(\frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y}) - rv_o + \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}$$

Here, we used: $\frac{\partial}{\partial x}(\frac{D_{*g}}{dt} v_o) - \frac{\partial}{\partial y}(\frac{D_{*g}}{dt} u_o) = D_{*gt}(\frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y}) + (\frac{\partial u_o}{\partial x} + \frac{\partial v_o}{\partial y})(\frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y})$.

Thus,

$$D_{*gt}(\frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y} + ry) = \frac{\partial w_1}{\partial z} + curl\mathbf{G} \qquad (5.1.14)$$

As in the shallow water case, the evolution of the O(1) variables u_o, v_o are determined by the $O(\varepsilon)$ variables. Since

$$w_1 = -\frac{1}{\Gamma_*} D_{*gt} \rho_o + \frac{S}{\Gamma_*} = -D_{*gt}(\frac{\rho_o}{\Gamma_*}) + \frac{S}{\Gamma_*} \qquad (5.1.15)$$

$$\partial_z w_1 = -D_{gt}[\frac{\partial}{\partial z}(\frac{\rho_0}{\Gamma_*})] + \frac{\partial}{\partial z} \partial(\frac{S}{\Gamma_*})$$

Thus, we have the nondimensional QG P.V. equation:

$$D_{*gt}\{ry + \frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y} + \frac{\partial}{\partial z}(\frac{\rho_o}{\Gamma_*})\} = curl\mathbf{G} + \partial_z(\frac{S}{\Gamma_*}) \qquad (5.1.16)$$

or in the dimensional form

$$D_g q = S_q \qquad (5.1.17a)$$

where

$$q = f_o + \beta y + \zeta + \partial_z \left(\frac{f_o \rho}{d\rho_s / dz} \right) \qquad (5.1.17b)$$

$$S_q = \frac{1}{\rho} \mathbf{k} \cdot \nabla \times \mathbf{F} + \partial_z \left(\frac{f_o S_0}{d\rho_s / dz} \right) \qquad (5.1.17c)$$

$$\zeta = \partial_x v_g - \partial_y u_g = \nabla^2{}_H \psi \qquad (5.1.17d)$$

$$D_g = \partial_t + u_g \partial_x + v_g \partial_y = \partial_t + J(\psi, \quad) \qquad (5.1.17e)$$

Since now

$$\rho = -\frac{\rho_m f_o}{g} \frac{\partial \psi}{\partial z}, \qquad (5.1.18)$$

$$N^2 = -\frac{g}{\rho_m} \frac{d\rho_s}{dz}, \qquad (5.1.19)$$

we have

$$\frac{\rho}{\frac{d\rho_s}{dz}} = \frac{f_o}{N^2} \frac{\partial \psi}{\partial z}$$

The dimensional QGPV equation can be written in terms of ψ as:

$$\partial_t q + J(\psi, q) = S_q \qquad (5.1.20a)$$

$$q = f_o + \beta y + \nabla^2{}_H \psi + \partial_z \left(\frac{f_o{}^2}{N^2} \frac{\partial \psi}{\partial z} \right) \qquad (5.1.20b)$$

For unforced, adiabatic flow, q is conserved along the geostrophic flow (which is different from the original 3-D flow!)

4. Application of QG dynamics: Diagnostic Perspective.

The QG dynamic framework is very useful for the diagnosis and understanding of the mechanism of various large scale circulations, from synoptic to planetary scales.

1) Planetary circulation and the generalized Sverdrup relation

Consider the special case where the horizontal scale much larger than the deformation radius and horizontal temperature advection is not important, such that the thermodynamic equation can be simplified as the balance between adiabatic ascending and diabatic heating, that is in (5.1), the thermodynamic equation

$$\partial_t \rho + (\vec{\mathbf{u}} \bullet \nabla)\rho + w\frac{d\rho_s}{dz} = S_0 = -\frac{\rho_m}{g}Q$$

(so that Q>0 represents heating) can be approximated as

$$w\frac{d\rho_s}{dz} = -\frac{\rho_m}{g}Q \qquad \text{or} \qquad wN^2 = Q \qquad (5.1.21)$$

where we have used $N^2 = -\frac{g}{\rho_m}\frac{d\rho_s}{dz} > 0$ for a stable stratification. This is equivalent to the O(1) thermodynamic equation (5.1.15) reduced to

$$w_1 = \frac{-Q}{\Gamma_*}$$

and therefore the QGPV in (5.1.20b) can be shown to reduce to the form of

$$q = f_o + \beta y + \nabla^2{}_H \psi$$

where the stretching effect between different isopycnals is negligible. Now, in the steady state, and neglect relative vorticity (very large scale), the QGPV equation (5.1.20) is reduced to a generalized Sverdrup relation (see Chapter 3, section 3.3):

$$\beta\psi_x = S_q = \frac{1}{\rho}\mathbf{k}\cdot\nabla\times\mathbf{F} + \partial_z\left(\frac{f_o S_0}{d\rho_s/dz}\right) \qquad (5.1.22)$$

In the absence of differential heating, and assume the momentum forcing is dominated by the wind stresss $\mathbf{F} = \boldsymbol{\tau}$, (5.1.22) reduces to the wind-driven oceanic Sverdrup relation (3.3.1) or (3.3.4) forced by the wind stress curl. For the atmosphere, assume the momentum forcing is negligible **F**=0, we have from (5.1.22) the Sverdrup relation forced by the differential heating

$$\beta\psi_x = \partial_z\left(\frac{f_o S_0}{d\rho_s/dz}\right) = \frac{1}{f_0}\partial_z\left(\frac{f_o{}^2 Q}{N^2}\right) = \left(\frac{f_o}{N^2}\right)\partial_z Q \qquad (5.1.23)$$

Therefore, for a deep heating such as latent heating, the lower layer flow is northward because $\partial_z Q > 0$ and upper layer flow is southward because $\partial_z Q < 0$. Unlike in the tropics, the meridional flow is no longer axis-symmetric to the heating forcing. If some Newtonian cooling is assumed (linear damping of density/temperature in the thermodynamic equation), one can show that the surface low pressure and descending motion are induced to the west of the heating, due to the westward propagation of planetary wave.

2) QG diagnosis and the generalized geopotential tendency equation.

Now, we derive diagnosis equation for geopotential tendency (pressure tendency) and the vertical motion in the QG framework. This is important for meterological synoptic dynamics and other purposes. We will use the dimensional form of stratified equations directly. The QG thermodynamic equation can be approximated from the original thermodynamic equation (5.1.1e) by replacing the horizontal wind as the geostrophic wind (5.10) and the density (or potential temperature in the atmospheric case, see next) using the hydrostatic equation (5.11) as:

$$\partial_t \left(\frac{\partial \phi}{\partial z} \right) + \vec{v}_g \bullet \nabla (\frac{\partial \phi}{\partial z}) + wN^2 = Q \qquad (5.1.24)$$

Here we have defined the geopotential height as

$$\phi = \psi / f_0, \qquad (5.1.25)$$

such that the hydrostatic equation (5.1.11) becomes

$$\frac{\partial \phi}{\partial z} = -\frac{g\rho}{\rho_m}, \text{ or } \rho = -\frac{\rho_m}{g}\frac{\partial \phi}{\partial z}, \qquad (5.1.26)$$

and

$$N^2 = -\frac{g}{\rho_m}\frac{d\rho_s}{dz}, \quad Q = -\frac{g}{\rho_m} S_0, \ \vec{v}_g \bullet \nabla a = J(\psi, a) \qquad (5.1.27)$$

The vorticity equation (the eqn. above (5.1.14)) can be written as the vorticity equation but with the voriticity and advection using the geostrophic wind

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g \bullet \nabla \right) (\nabla^2 \psi + \beta y) = -f_0 (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + curl\boldsymbol{\tau}$$

Or using geopotential height and continuity equation as

$$\frac{\partial \nabla^2 \phi}{\partial t} + f_0 \mathbf{v}_g \bullet \nabla(\frac{1}{f_0}\nabla^2\phi + \beta y) = f_0^{\ 2}\frac{\partial w}{\partial z} + f_0 curl\boldsymbol{\tau} \qquad (5.1.28)$$

Define the tendency for geopotential height as

$$\chi = \frac{\partial \phi}{\partial t}$$

The thermodynamic equation (5.1.24) and the vorticity equation (5.1.28) become

$$\frac{\partial \chi}{\partial z} = -\vec{v}_g \bullet \nabla(\frac{\partial \phi}{\partial z}) - N^2 w + Q \qquad (5.1.29a)$$

$$\nabla^2 \chi = -f_0 \mathbf{v}_g \bullet \nabla(\frac{1}{f_0}\nabla^2\phi + \beta y) + f_0^{\ 2}\frac{\partial w}{\partial z} + f_0 curl\boldsymbol{\tau} \qquad (5.1.29b)$$

These two equations form another set of QG equation: The first equation states that the vertical derivative of the geopotential tendency is equal to the sum of the thickness advection, adiabatic thickness change due to vertical motion and external heating. The second equation indicates that the horizontal Laplacian of th egeopotential tendency is equal to the sume of the vorticity advection, the vorticity generation by the divergence and curl of external forcing. These two equations can be used to derive diagnosis equations for geopotential tendency and omega using the geopotential height of the same time only. First for the geopotential equation, if we operate

$$f_0^{\ 2}\frac{\partial}{\partial z}\frac{1}{N^2}$$

On the thermodynamic equation (5.1.29a), and then sum it with the vorticity equation (5.1.29b), we derive a diagnostic equation for geopotential tendency as

$$\boxed{\begin{aligned}\left[\nabla^2 + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\right)\right]\chi &= f_0\left[-\mathbf{v}_g \bullet \nabla(\frac{1}{f_0}\nabla^2\phi + \beta y) + curl\boldsymbol{\tau}\right] \\ &+ \frac{\partial}{\partial z}\left[-\frac{f_0^2}{N^2}\vec{v}_g \bullet \nabla(\frac{\partial \phi}{\partial z}) + \frac{f_0^2}{N^2}Q\right]\end{aligned}} \qquad (5.1.30)$$

This states that the change of geopotential tendency is the sum of the effective vorticity forcing, which consists of the advection of absolute vorticity and curl stress and the differential effective heating, which consists of the temperature advection and diabatic heating. Note here, the advection of absolute vorticity is equivalent to a curl tau forcing, , both contributing to an effective vorticity forcing; the vertical shear of temperature advection is equivalent to the differential heating, both contributing to an effective differential heating. This more generalized view will be seen helpful for our intuitive thinking.

To further understand the utility of the tendency equation, for a wave of given horizontal structure for all the terms,

$$\chi \sim -\mathrm{X}(z)e^{i(kx+ly-\omega t)},$$

$$f_0\left[-\mathbf{v}_g \bullet \nabla(\frac{1}{f_0}\nabla^2\phi + \beta y) + curl\boldsymbol{\tau}\right] = F_v(z)e^{i(kx+ly-\omega t)}$$

$$-\frac{f_0^2}{N^2}\vec{v}_g \bullet \nabla(\frac{\partial\phi}{\partial z}) + \frac{f_0^2}{N^2}Q = F_T(z)e^{i(kx+ly-\omega t)},$$

Assuming N constant, the geopotential equation gives

$$\frac{d^2\mathrm{X}}{dz^2} - \frac{\mathrm{X}}{h_R^2} = \frac{N^2}{f_0^2}(F_v - \frac{dF_T}{dz}) \quad (5.1.31)$$

with

$$h_R^2 = \frac{f_0^{\ 2}}{N^2(k^2+l^2)} \sim \left(\frac{f_0 L}{N}\right)^2 , \quad (5.1.32)$$

L is the horizontal scale of the disturbance and h_R is called the Rossby height. (also see exercise E5.1) (This is the reverse of internal deformation radius, that is, for a given depth of a structure of h_R, the corresponding internal deformation radius is $L_I=Nh_R/f_0$. See discussion in the next section). (5.1.31) states that if there is an anomalous effective vorticity forcing F_v or effective differential heating F_{Tz} at the depth z_1 with a horizontal scale of L, the response of the geopotential height (pressure) response will be limited to a depth of Rossby height from the forcing height. Therefore, effectively, the Rossby height represents the vertical penetration of a disturbance on the pressure field. For developing synoptic waves with westward tilt of trough/ridge line (see later chapter 6), on the ridge/trough line, in the geopotential tendency equation (5.1.30), vorticity advection is small and the development of the geopotential tendency is determined by the low level temperature advection (assuming diabatic heating is weak at short synoptic time scales), the low level cold (warm) advection towards the trough (ridge) then further intensifies the upper level trough/ridge. (see figure below).

$$\left[\nabla^2 + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\right)\right]\chi \sim -\chi \approx \frac{\partial}{\partial z}\left[\frac{f_0^2}{N^2}Q_{eff}\right] = \frac{\partial}{\partial z}\left[-\frac{f_0^2}{N^2}\vec{v}_g \bullet \nabla(\frac{\partial \phi}{\partial z})\right]$$

Now, the effective heating is simply the temperature advection

$$Q_{eff} = -\vec{v}_g \bullet \nabla(\frac{\partial \phi}{\partial z}).$$

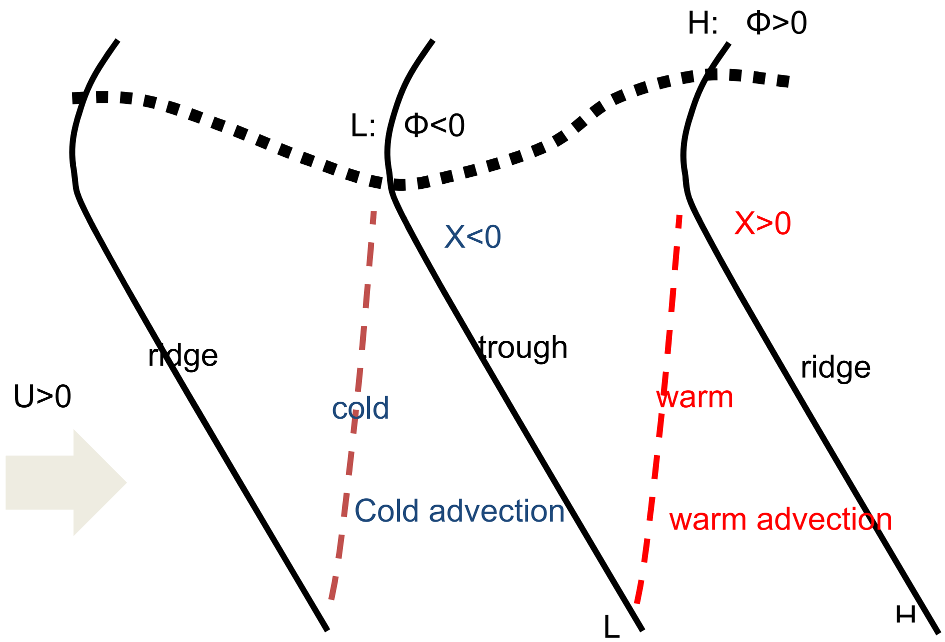


Figure: developing synoptic disturbances in the westerly

In the other extreme limit of very large scale diabatic heating Q, such that relative vorticity and temperature advection become negligible, the geopotential tendency equation can be simplified as

$$\left[\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\right)\right]\chi = -f_0 v_g \beta + \frac{\partial}{\partial z}\left[\frac{f_0^2}{N^2}Q\right]. \quad (5.1.32)$$

Or approximately

$$-\chi \sim -f\beta v + \frac{\partial Q}{\partial z}. \quad (5.1.33)$$

First, if $\beta = 0$, we have

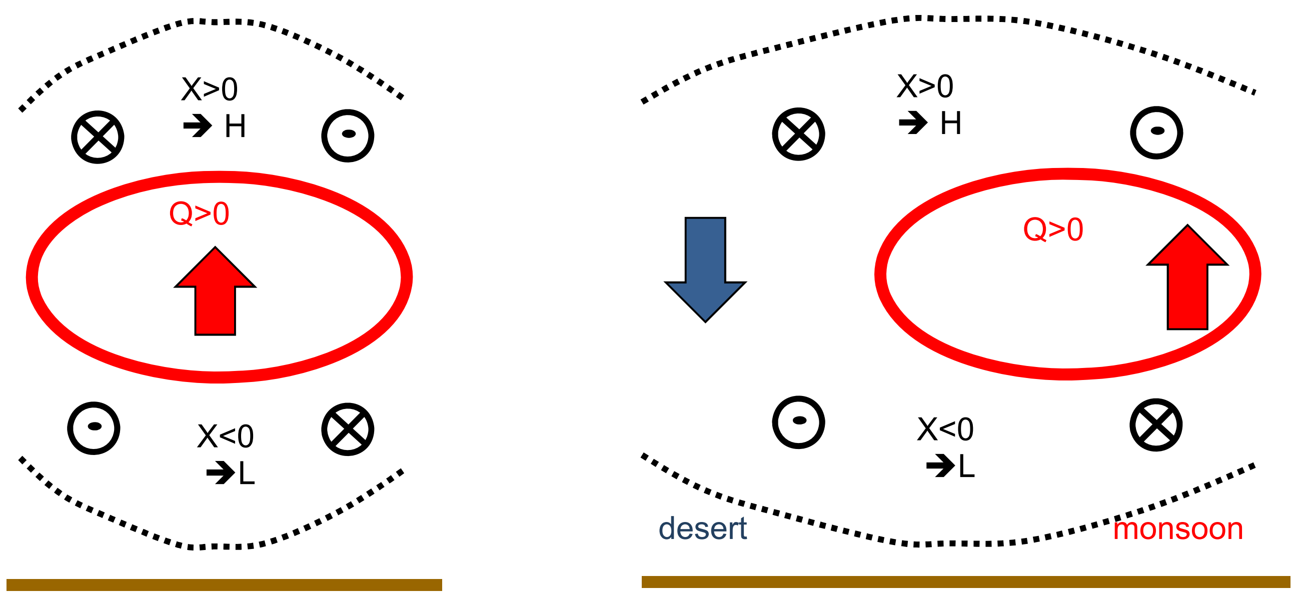
$$-\chi \sim +\frac{\partial Q}{\partial z} \quad (5.1.34)$$

So, below the heating $\frac{\partial Q}{\partial z} > 0$, low pressure develops $\chi < 0$, and above the heating a high pressure develops (see Fig.5.x)

` In general, for large scale flow, $\beta > 0$, (5.1.34) indicates that the low pressure and high pressure center will not be collocated with the heating center. Indeed, Assuming N=const, and using geostrophy, (5.1.32) can be written in terms of geostrophic streamfunction (or geopotential height) as

$$(\partial_t + \varepsilon)\psi_{zz} + \beta \frac{N^2}{f_0^2}\psi_x = +\frac{1}{f_0}\frac{\partial Q}{\partial z} \qquad (5.1.35)$$

Here, we have included a thermal damping term ε with the local variability, such that the forced response can reach a final steady state. If we neglect the first term of geopotential tendency, this reduces to thermally driven Sverdrup relation (5.1.23). Now, with the geopotential tendency, on a beta-plane, one can show (E5.??) that, below the heating $\frac{\partial Q}{\partial z} > 0$, low pressure develops $\chi < 0$ and extends westward, and above the heating a high pressure develops that extends westward. The westward extension is due to the planetary Rossby wave and the final state is achieved by thermal damping.



β=0 case β>0 case

Fig.5.x Response to large scale deep diabatic heating on a (left) f-plane and (right) beta-plane. The pressure field is diagnosed from the geopotential tendency equation while the vertical motion from the omega-equation (see later discussion).

3) QG diagnosis and the generalized omega-equation.

Now, we turn to the vertical motion and derive the so-called omega-equation. We first take Laplacian to (5.1.29a) and vertical derivative to (5.1.29b) as

$$\nabla^2 \frac{\partial \chi}{\partial z} = -\nabla^2[\vec{v}_g \bullet \nabla(\frac{\partial \phi}{\partial z})] - N^2 \nabla^2 w + \nabla^2 Q$$

$$\frac{\partial}{\partial z}(\nabla^2 \chi) = -f_0 \frac{\partial}{\partial z}\left[\mathbf{v}_g \bullet \nabla(\frac{1}{f_0}\nabla^2\phi + \beta y) - curl\boldsymbol{\tau}\right] + f_0^{\ 2}\frac{\partial^2 w}{\partial z^2}$$

Subtraction of the two equations give a diagnostic equation for vertical motion (omega-equation) as

$$\boxed{\begin{aligned}\left(\nabla^2 + \frac{f_0^{\ 2}}{N^2}\frac{\partial^2}{\partial z^2}\right)w = -\frac{f_0}{N^2}\frac{\partial}{\partial z}\left[-\mathbf{v}_g \bullet \nabla(\frac{1}{f_0}\nabla^2\phi + \beta y) + curl\boldsymbol{\tau}\right] \\ + \frac{1}{N^2}\nabla^2\left[-\vec{v}_g \bullet \nabla(\frac{\partial\phi}{\partial z}) + Q\right]\end{aligned}} \quad (5.1.36)$$

The vertical motion can now be diagnosed from the geopotential field of the same time. For a wave structure $w \sim \cos(mz)e^{i(kx+ly-\omega t)}$, we have roughly

$$\left(\nabla^2 + \frac{f_0^{\ 2}}{N^2}\frac{\partial^2}{\partial z^2}\right)w \sim -(k^2 + l^2 + \frac{f_0^{\ 2}}{N^2}m^2)w \sim -w$$

$$\nabla^2\left[-\vec{v}_g \bullet \nabla(\frac{\partial\phi}{\partial z})\right] \sim \vec{v}_g \bullet \nabla(\frac{\partial\phi}{\partial z}).$$

Thus, the omega-equation becomes

$$(k^2 + l^2 + \frac{f_0^{\ 2}}{N^2}m^2)w \sim \frac{f_0}{N^2}\frac{\partial}{\partial z}\left[-\mathbf{v}_g \bullet \nabla(\frac{1}{f_0}\nabla^2\phi + \beta y) + curl\boldsymbol{\tau}\right] - \frac{1}{N^2}\nabla^2\left[-\vec{v}_g \bullet \nabla(\frac{\partial\phi}{\partial z}) + Q\right]$$

.

So the geopotential tendency and vertical motion can be diagnosed directly from the weather map of geopotential height. Of course, high order derivatives are used so it is still difficult to assess those terms accurately in the observation. Nevertheless, these equations shed light on the mechanisms of the geopotential tendency and vertical motion.

For a typical developing synoptic system that tilts westward (below Fig.5.x), above the surface low pressure center, upper air advects positive vorticity, there is a rising motion, and above the surface high, the upper air advects negative vorticity, which leads to a descending. In addition, to the west of the surface low, the cold advection leads to descending, and to the east of the surface low, the warm advection leads to an ascending. As a result, the strongest ascending for a developing synoptic system occurs east of the surface low pressure center.

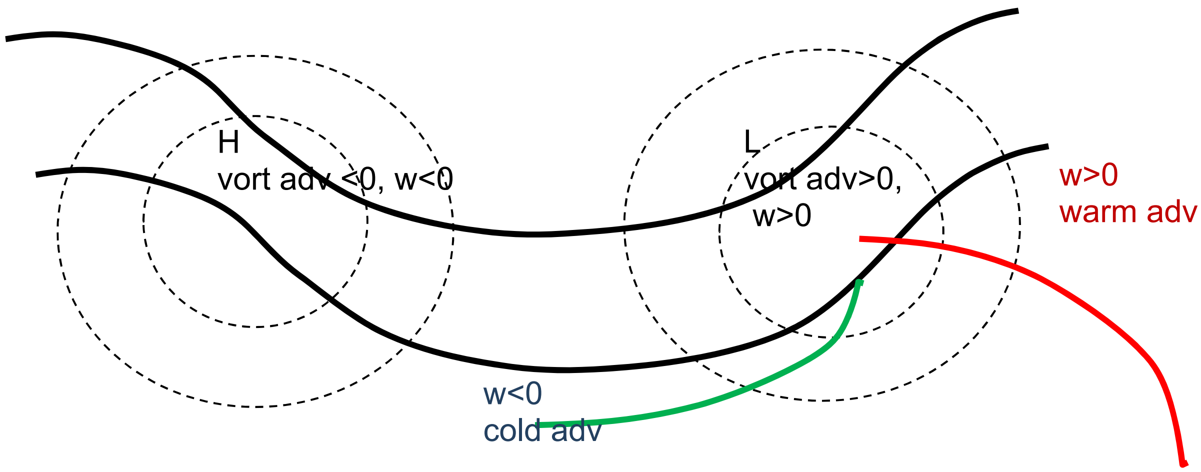


Fig. 5.x: vertical motion diagnosizd from omega-equation for a developing synoptic system.

In the other limit of large scale heating, we can neglect relative vorticity and temperature advection, the omega equation gives

$$\frac{f_0}{N^2} w \sim -\frac{f_0}{N^2} \beta v_z - \frac{1}{N^2} \nabla^2 Q \qquad (5.1.37)$$

If β=0, we have simply

$$w \sim -\nabla^2 Q \sim +(k^2 + l^2) Q \sim Q \qquad (5.1.38)$$

So, the air rises with heating (thermodynamic balance between diabatic heating and adiabatic ascending and vice versa. However, for $\beta > 0$, there will be an ascending induced by the northward shear $v_z > 0$ and vice versa, Since for planetary scale diabatic heating, the pressure field is shifted westward as shown before in geopotential tendency equation (5.1.35) and Fig.5x(right), the final ascending motion will be shifted to the eastern part of the heating center. In the mean time, there will also be a descending to the west of the heating region. This is the so called monsoon-desert mechanism (Rodwell and Hoskins, 1996, JAS). The planetary scale geopotential field and vertical motion can be understood from the generalized Sverdrup relation as discussed before. This vertical motion will also be derived in exercise E5.??.

Sec5.1: Appendix. The Atmosphere Case

The QGPV equation in the atmosphere can be derived parallel to the oceanic equation. Typical scales are chosen as

$$u,\ v \sim U, \quad x,\ y \sim L,\ \ w \sim U\frac{D}{L}, \quad z \sim D,\ \ t \sim \frac{L}{U},$$

the Rossby number is $\varepsilon = \frac{U}{f_o L}$, the potential temperature is written as

$$\theta(x,y,z,t) = \Theta(z) + \theta'(x,y,z,t)$$

and the geostrophic potential high as

$$\phi(x,y,z,t) = \Phi(z) + \phi'(x,y,z,t)$$

where

$$\frac{d\Phi}{dz} = g\frac{\Theta(z)}{\Theta_s}$$

$$\frac{d\phi'}{dz} = g\frac{\theta'}{\Theta_s}$$

For large scale flows, the geopotential height anomaly is scaled as

$$\phi \sim f_o LU$$

such that the pressure gradient is comparable to the Coriolis force at the first order. The source/sink is also weak such that

$$\frac{1}{f_o U\rho}\mathbf{F} = \varepsilon\Gamma\ , \qquad \text{where } |G| \le O(1).$$

The local β-plane is adopted as

$$\frac{\beta L}{f_o} = \gamma\varepsilon\ ,\ \ O(r) \sim 1.$$

The two momentum equations are represented in dimensionless variables as:

$$\varepsilon\{\frac{\partial u_*}{\partial t_*} + (\mathbf{u}_* \bullet \nabla_*)u_* - \gamma y_* v_*\} - v_* = -\frac{\partial \phi_*}{\partial x_*} + \varepsilon G_x$$

$$\varepsilon\{\frac{\partial v_*}{\partial t_*} + (\mathbf{u}_* \bullet \nabla_*)v_* + \gamma y_* u_*\} + u_* = -\frac{\partial \phi_*}{\partial y_*} + \varepsilon G_y$$

The mass equation is

$$\frac{\partial u_*}{\partial x_*} + \frac{\partial v_*}{\partial y_*} + \frac{1}{p}\frac{\partial}{\partial z_*}(pw_*) = 0$$

The scale of θ' is estimated from the hydrostatic equation

$$\frac{\partial \phi'}{\partial z} = g\frac{\theta'}{\Theta_s}$$

This gives

$$\theta' \sim \frac{\phi}{gD}\Theta_s \sim \frac{f_o L U}{gD}\Theta_s \sim \Theta_s(\frac{L}{L_D})^2\varepsilon = \Gamma$$

where $L_D{}^2 = \frac{gD}{f_o^2}$ is the external deformation radius Thus,

$$\theta' \sim \Gamma\theta_* \qquad \text{where } \theta_* \sim O(1),$$

and the hydrostatic balance is

$$\frac{\partial \phi_*}{\partial z_*} = \theta_*$$

With these scaling, the thermodynamic equation becomes

$$\frac{U\Gamma}{L}\{\frac{\partial \theta_*}{\partial t_*} + (\mathbf{u}_* \bullet \nabla_*)\theta_*\} + \frac{UD}{L}w_*\frac{d\Theta}{dz} = Q_a$$

$$\Rightarrow \varepsilon\{\frac{\partial \theta_*}{\partial t_*} + (\mathbf{u}_* \bullet \nabla_*)\theta_*\} + \frac{UD}{f_o L\Gamma}w_*\frac{d\Theta}{dz} = \frac{Q_a}{f_o\Gamma}$$

The scale of $\frac{d\Theta}{dz}$ can be anticipate to satisfy $\frac{d\Theta}{dz} \geq \frac{1}{\varepsilon}\frac{\partial \theta'}{\partial z} >> \frac{\partial \theta}{\partial z}$. For adiabatic flows, the solution is always along isentropics, $\frac{d\theta}{dt} = 0$. Since QG equations require $\frac{w}{u} \leq \varepsilon\frac{D}{L}$, the slope of the isentropic surface must be $\leq \varepsilon\frac{D}{L}$

$$\frac{\partial \theta}{\partial x} \Big/ \frac{\partial \theta}{\partial z} \cong \alpha \leq \varepsilon\frac{D}{L}$$

Since $\frac{\partial \Theta}{\partial x} = 0$, we have

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta'}{\partial x} \sim \frac{\partial \theta'}{\partial z}\frac{D}{L}$$

and therefore

$$\frac{\partial \theta'}{\partial z} \Big/ \frac{\partial(\Theta+\theta')}{\partial z} \le \varepsilon$$

$$\frac{\partial \theta'}{\partial z} \Big/ \frac{\partial(\Theta)}{\partial z} \le \varepsilon$$

Thus, we are led to the scale of

$$\frac{d\Theta}{dz} = \frac{\Gamma}{D\varepsilon}\Gamma_*(z) \qquad \text{where } \Gamma_*(z) \le O(1).$$

If we define a buoyancy frequency *N(z),* then

$$N^2 = \frac{g}{\Theta_s}\frac{d\Theta}{dz}$$

The thermodynamic equation becomes

$$\varepsilon\{\frac{\partial \theta_*}{\partial t_*} + (\mathbf{u}_* \nabla_*)\theta_*\} + w\Gamma_* = \varepsilon Q$$

where $\varepsilon Q = \dfrac{Q_a}{f_o \Gamma}$ and $Q \le O(1)$ is consistent with leading order adiabatic.

The complete set of dimensionless equations are now (drop *):

$$v - \frac{\partial \phi}{\partial x} = \varepsilon\{\frac{\partial u}{\partial t} + (\mathbf{u} \bullet \nabla)u - ryv - G_x\}$$

$$-u - \frac{\partial \phi}{\partial y} = \varepsilon\{\frac{\partial v}{\partial t} + (\mathbf{u} \bullet \nabla)v + ryu - G_y\}$$

$$-w\Gamma_* = \varepsilon\{\frac{\partial \theta}{\partial t} + (\mathbf{u} \bullet \nabla)\theta - Q\}$$

$$\frac{\partial \phi}{\partial z} - \theta = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{p}\frac{\partial}{\partial z}(pw) = 0$$

The variables will be expanded as follows:

$$u = u_o + \varepsilon u_1 + O(\varepsilon^2)$$

$$v = v_o + \varepsilon v_1 + O(\varepsilon^2)$$

$$\phi = \phi_o + \varepsilon \phi_1 + O(\varepsilon^2)$$

$$\theta = \theta_o + \varepsilon \theta_1 + O(\varepsilon^2)$$

$$w = w_o + \varepsilon w_1 + O(\varepsilon^2)$$

The O(1) equations are

$$v_o = \frac{\partial \phi_o}{\partial x}$$
$$u_o = -\frac{\partial \phi_o}{\partial y}$$
$$w_o = 0$$
$$\frac{\partial \phi}{\partial z} = \theta_o$$
$$\frac{\partial u_o}{\partial x} + \frac{\partial v_o}{\partial y} + \frac{1}{p}\frac{\partial}{\partial z}(p w_o) = 0$$

In dimensional form, the geostrophic balance is:

$$v_g = \frac{1}{f_o}\frac{\partial \phi'}{\partial x} \equiv \partial_x \psi$$
$$u_g = -\frac{1}{f_o}\frac{\partial \phi'}{\partial y} \equiv -\partial_y \psi$$

where $\xi = \frac{\phi'}{f_o}$ is the geostrophy stream function. The hydrostatic equation is $\frac{\partial \phi'}{\partial z} = \theta'$ or $\frac{\theta'}{\Theta_s} = \frac{f_o}{g}\frac{\partial \psi}{\partial z}$. This leads to the thermal wind relationship

$$\frac{\partial v_g}{\partial z} = \frac{\partial^2 \psi}{\partial x \partial z} = \frac{g}{f_o \theta_s}\frac{\partial \theta'}{\partial x}, \quad \frac{\partial u_g}{\partial z} = -\frac{\partial^2 \psi}{\partial y \partial z} = -\frac{g}{f_o \Theta_s}\frac{\partial \theta'}{\partial y}$$

At the next order, we have

$$v_1 - \frac{\partial \phi_1}{\partial x} = \frac{D_{*g}}{Dt} u_o - ryv_o - G_x$$
$$-u_1 - \frac{\partial \phi_1}{\partial y} = \frac{D_{*g}}{Dt} v_o + ryu_o - G_y$$
$$-w_1 \Gamma_* = \frac{D_{*g}}{Dt}\theta_o - Q$$
$$\frac{\partial \phi}{\partial z} - \theta = 0$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{1}{p}\frac{\partial (p w_1)}{\partial z} = 0$$

where $\frac{D_{*g}}{Dt} = \frac{\partial}{\partial t} + u_o \frac{\partial}{\partial x} + v_o \frac{\partial}{\partial y}$. The vorticity equation can be derived as:

$$D_{*gt}(\frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y} + ry) = \frac{1}{p}\frac{\partial}{\partial z}(pw_1) + curl\vec{G}$$

Since

$$w_1 = -\frac{1}{\Gamma_*} D_{*gt}\theta_o + \frac{Q}{\Gamma_*} = -D_{*gt}(\frac{\theta_o}{\Gamma_*}) + \frac{Q}{\Gamma_*}$$

$$\frac{1}{p}\partial_z(pw_1) = -D_{gt}[\frac{1}{p}\frac{\partial}{\partial z}(p\frac{\theta_0}{\Gamma_*})] + \frac{1}{p}\frac{\partial}{\partial z}\partial(p\frac{Q}{\Gamma_*})$$

we have the nondimensional QG P.V. equation:

$$D_{*gt}\{ry + \frac{\partial v_o}{\partial x} - \frac{\partial u_o}{\partial y} + \frac{1}{p}\frac{\partial}{\partial z}(\frac{p\theta_o}{\Gamma_*})\} = curlG + \frac{1}{p}\partial_z(\frac{pQ}{\Gamma_*})$$

In the dimensional form

$$D_g q = S_q$$

where

$$q = f_o + \beta y + \zeta + \frac{1}{p}\partial_z(\frac{pf_o\theta}{d\Theta/dz}), \quad S_q = \frac{1}{\rho}\mathbf{k}\cdot\nabla\times\mathbf{F} + \frac{1}{p}\partial_z(\frac{pf_oQ_a}{d\Theta/dz})$$

$$\zeta = \partial_x v_g - \partial_y u_g = \nabla^2{}_H\psi, \qquad D_g = \partial_t + u_g\partial_x + v_g\partial_y$$

Now, with

$$\theta' = \frac{\theta_s f_o}{g}\frac{\partial\psi}{\partial z}, \; N^2 = \frac{g}{\Theta_s}\frac{d\Theta}{dz}$$

and therefore $\frac{\theta'}{d\Theta/dz} = \frac{f_o}{N^2}\frac{\partial\psi}{\partial z}$, we have the QGPV equation for the atmosphere as:

$$\boxed{\partial_t q + J(\psi, q) = S_q} \qquad (5.1.A1)$$

$$\boxed{q = f_o + \beta y + \nabla^2{}_H\psi + \frac{1}{p}\partial_z(\frac{pf_o{}^2}{N^2}\frac{\partial\psi}{\partial z})}. \qquad (5.1.A2)$$

Sec. 5.2: Rossby Waves in Stratified Flows

1. Dispersion Relationship

As in the shallow water case, we study small perturbations linearized on a mean zonal flow. The linearized QGPV equation is:

$$(\partial_t + U\partial_x)q' + v'Q_y = 0$$

where the mean and perturbation potential vorticity are

$$Q_y = \beta - \partial_{yy}U - \frac{1}{p}\partial_z\left[\frac{pf_o^2}{N^2}\partial_z U\right],$$

$$q' = \partial_{xx}\psi + \partial_{yy}\psi + \frac{1}{p}\partial_z\left[\frac{pf_o^2}{N^2}\frac{\partial\psi}{\partial z}\right]$$

We have used the atmospheric equation (5.1.21) and the oceanic equation can be recovered simply by setting $p=p_0$. The QGPV equation can be written in the perturbation streamfuction ψ as

$$(\partial_t + U\partial_x)\{\partial_{xx}\psi + \partial_{yy}\psi + \frac{1}{p}\partial_z\left[\frac{pf_0^2}{N^2}\partial_z\psi\right]\} + \partial_x\psi Q_y = 0 \qquad (5.2.1)$$

We will assume the basic state is slowly varying, that is the wave length in y, z directions are short relative to the scales at which U, N^2 and Q_y vary. (But, we don't need to assume *p(z)* slowly varying!). Assuming the solution of the form

$$\psi(x,y,z,t) = \mathrm{Re}\left[\Psi(y,z)e^{i\theta + \frac{z}{2H}}\right]$$

where

$$\theta = k(x-ct) + \int^y l(y')dy' + \int^z m(z')dz'$$

We have approximately

$$\partial_{xx}\psi \approx -k^2\psi \quad , \quad \partial_{yy}\psi \approx -l^2\psi$$

and

$$\frac{1}{p}\partial_z\left(\frac{pf_o^2}{N^2}\partial_z\psi\right) \cong \frac{f_o^2}{N^2}\frac{1}{p}\partial_z(p\partial_z\psi)$$

$$\approx \frac{f_o^2}{N^2}e^{\frac{z}{H}}\partial_z(e^{-\frac{z}{H}}\partial_z\psi)$$

$$\approx -\frac{f_o^2}{N^2}(m^2+\frac{1}{4H^2})\Psi e^{i\theta}e^{\frac{z}{2H}}$$

$$= -\frac{f_o^2}{N^2}(m^2+\frac{1}{4H^2})\psi$$

Therefore, (5.2.1) gives the dispersion relationship as

$$(U-c)[k^2+l^2+\frac{f_o^2}{N^2}(m^2+\frac{1}{4H^2})]-Q_y=0.$$

That is

$$c = U - \frac{Q_y}{k^2+l^2+\frac{f_o^2}{N^2}(m^2+\frac{1}{4H^2})} \qquad (5.2.2)$$

for the atmosphere. For the ocean, we can set H infinitely large (incompressible), so that

$$c = U - \frac{Q_y}{k^2+l^2+\frac{f_o^2}{N^2}m^2} \qquad (5.2.3)$$

The relation between the shallow water Rossby waves and the baroclinic Rossby waves here are readily seen if we make

$$L^2_{Dm} = \frac{N^2}{f^2(m^2+\frac{1}{4H^2})} \qquad (5.2.4)$$

The dispersion relationship can be put exactly the same form as the shallow water Rossby wave

$$c = U - \frac{Q_y}{k^2+l^2+L_{Dm}^{-2}}$$

The L_{Dm} is the deformation radius for baroclinic flows with a vertical wave number m: Since $L_{Dm} \sim 1/m$, the deformation radius increases and the wave speed faster for smaller m (or larger vertical scale), and vise versa. For typical atmospheric stratification, we have $L_{D1} \sim 1000\ km$, while for typical oceanic stratification, we have $L_{D1} \sim 50\ km$. In the limit of long wave $L >> L_{Dm}$ and in the absence of no mean flow, the dispersion relation is reduced to $c = -\beta L_{Dm}^2$. This has been derived directly in the homework E4.1.

2. Group Velocity and Vertical Propagation

For a given *k, l*, the dispersion relationship gives the vertical wave number

$$m^2 = \frac{N^2}{f_o^2}\left[\frac{Q_y}{U-c} - k^2 - l^2\right] - \frac{1}{4H^2} \quad (5.2.5)$$

When the RHS > 0, *m* is real, and the Rossby wave propagate vertically; when the RHS < 0, *m* becomes imaginary, and the waves are trapped vertically.

For propagation waves (real m), the group velocity can be calculated as

$$C_{gx} = \frac{\partial\omega}{\partial k} = U + \Delta[k^2 - l^2 - \frac{f_o^2}{N^2}(m^2 + \frac{1}{4H^2})]$$
$$C_{gy} = \frac{\partial\omega}{\partial l} = 2kl\Delta \quad (5.2.6)$$
$$C_{gz} = \frac{\partial\omega}{\partial m} = 2\frac{f_o^2}{N^2}km\Delta$$

where

$$\Delta = Q_y[k^2 + l^2 + \frac{f_o^2}{N^2}(m^2 + \frac{1}{4H^2})]^{-2} = \frac{(U-c)^2}{Q_y} \quad (5.2.7)$$

Note 1: If *l* is replaced by $\frac{f_o^2}{N^2}m$, we have C_{gy} the same as C_{gz} Therefore, mathematically, the y direction and z direction are very similar for Rossby waves. However, later, we will see that the physical meaning of the group velocity in these two directions differ dramatically.

In the case of *U=0*, we have $Q_y = \beta > 0$, we have $C_{px} = \frac{\omega}{k} < 0$, so the wave always propagates westward. Define the phase velocity as

$$\mathbf{C}_p = (C_{px}, C_{py}, C_{pz}) = \frac{\omega}{|\vec{K}|}\vec{K}_0 = \frac{\omega}{|\vec{K}|^2}\vec{K}$$

where $\vec{K}_0 = \vec{K} / |\vec{K}|$ is the unit vector in the direction of wave vector.

We see from (5.2.2) and (5.2.6) that (take the sign of *k* as positive, so ω =kc is negative)

$$sign(C_{py}) = -sign(l) \ , \quad \text{but} \quad sign(C_{gy}) = +sign(l)$$

$$sign(C_{pz}) = -sign(m), \quad \text{but} \quad sign(C_{gz}) = +sign(m)$$

Thus, the phase and group velocity are in the opposite directions.

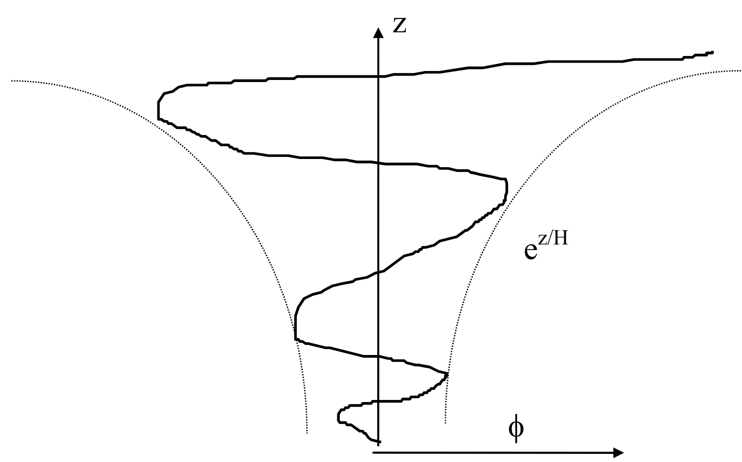
It should be pointed out that in the atmosphere, even if m is real (when the basic state allows), ψ does not vary with height simply as e^{imz}; instead, its amplitude increases with height as

$$\psi \sim e^{imz} e^{\frac{z}{2H}}$$

or the energy increases with height as

$$|\psi|^2 \sim e^{z/H} \sim 1/p$$

The amplification of the streamfunction with height is caused by the reduction of atmospheric density.



Finally, for stationary forcing (topography or large scale heating/cooling) *c=0*. Eqn.(5.2.5) shows that only those largest scale waves (smaller *k, l*) can propagate vertically in the westerly wind (*U>0*) (real m). This has been used to explain the observed stratosphere. Stratosphere disturbances are believed to originate from the troposphere. Observations show that the mid- and high latitude stratosphere is dominated by disturbances at planetary scales (wave number 1, 2 3), although the most energetic disturbances in the troposphere are at higher wave numbers (6 and 7). This is because only those very long waves can propagate into the stratosphere according to (5.2.2).

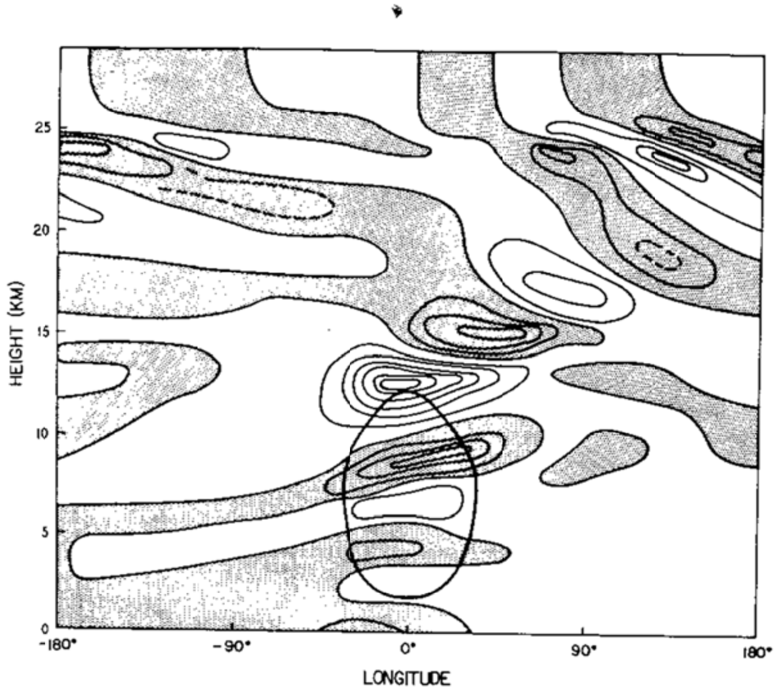


Fig. 11.13. Longitude–height section of the meridional velocity perturbation at the equator as found by Holton (1972, Fig. 9) for an antisymmetric source of diabatic heating that oscillates with an amplitude exceeding 4 K day^{-1} inside the heavy line. Contours are at 2 m s^{-1} intervals. The waves produced are mainly mixed planetary–gravity waves. The mean wind varies with height with a maximum eastward velocity of 8 m s^{-1} at 21 km, zero velocity at 25 km, and westward velocity above that level.

Fig.5.1: Vertical propagation of atmospheric waves

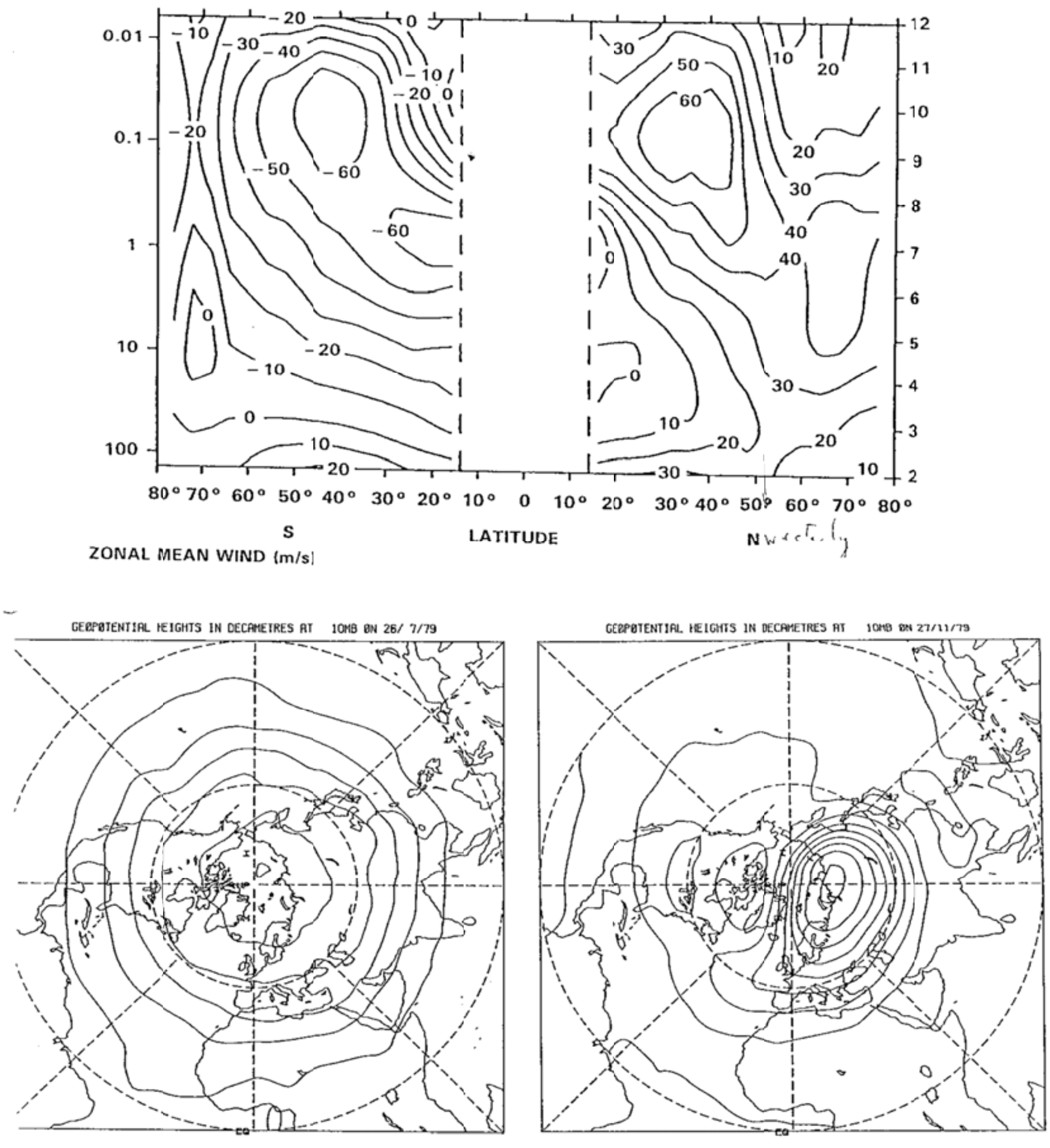
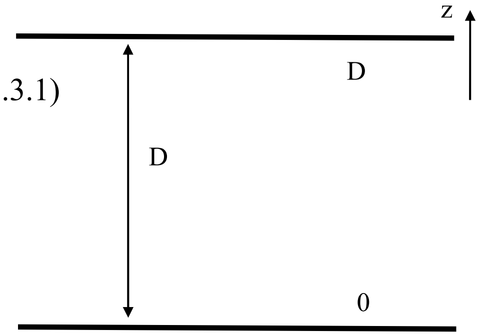


Fig.5.2: Geopotential height anomaly in the stratosphere

Sec. 5.3 Vertical Normal Modes

1: Vertical Modes in the Ocean

Consider an ocean with $U=0$ and N uniform. The mean PV gradient is therefore $Q_y = \beta$. The linearized QGPV equation is:



$$\partial_t[\partial_{xx} + \partial_{yy} + \partial_z(\frac{f_o^2}{N^2}\partial_z)]\psi + \beta\partial_x\psi = 0 \quad (5.3.1)$$

On the bottom (assume flat)

$$w(x,y,0,t) = 0 \quad (5.3.2)$$

At the top $z=D+\eta(x,y,t)$

We have a rigid lid

$$w(x,y,D,t) = 0. \quad (5.3.3)$$

To write the vertical boundary condition in terms of ψ, we resort to the thermodynamic equation:

$$\partial_t\rho' + w\frac{d\rho_s}{dz} = 0$$

The hydrostatic balance gives:

$$\frac{\partial\psi}{\partial z} = -\frac{g}{f_o}\frac{\rho'}{\rho_m} \quad (5.3.4)$$

We then have

$$w = -\frac{\partial_t\rho'}{\frac{d\rho_s}{dz}} = \frac{f_o\rho_m}{g\frac{d\rho_s}{dz}}\frac{\partial^2\psi}{\partial t\partial z} = -\frac{f_o}{N^2}\partial_{tz}\psi \quad (5.3.5)$$

Thus, the vertical boundary conditions (5.3.2) and (5.3.3) become

$$\partial_{tz}\psi = 0 \quad \text{at } z=0,\ D \quad (5.3.6)$$

Note 1: Effect of bottom topography at $z_B(x,y)$ is

$$w[x,y,z_B(x,y)] = \vec{u}(x,y,z_B)\bullet\nabla z_B = J[\psi(x,y,z_B), z_B].$$

In the special case of a linear meridional slope of small amplitude $z_B = \Lambda y$, the bottom boundary condition becomes simply $w[x,y,z_B(x,y)] = \Lambda\partial_x\psi(x,y,z_B) \approx \Lambda\partial_x\psi(x,y,0)$.|||

Returning to the QGPV equation. We look for separable solutions of the form

$$\psi = \phi(z)\Psi(x,y)e^{-i\omega t}$$

Substitute this into the QGPV equation, we have the equation for the vertical structure as

$$\frac{d}{dz}\left[\frac{f_o^{\ 2}}{N(z)^2}\frac{d\phi}{dz}\right] = -\lambda^2\phi \tag{5.3.7}$$

and the equation for the horizontal structure as

$$-i\omega\left[\partial_{xx}\Psi + \partial_{yy}\Psi - \lambda^2\Psi\right] + \beta\frac{\partial\Psi}{\partial x} = 0 \tag{5.3.8}$$

The vertical boundary conditions (5.3.6) becomes

$$\frac{d\phi}{dz} = 0 \quad \text{on } z=0,D \tag{5.3.9}$$

Therefore, (5.3.7) and (5.3.9) form an eigenvalue problem. The eigenvalues are real.

Indeed, notice (5.3.9), $\int_0^D \phi\frac{N^2}{f_o^{\ 2}} \times (5.3.7)$ gives

$$-\lambda^2\int_0^D \frac{N^2}{f_o^{\ 2}}\phi^2 dz = \int_0^D \phi\frac{d^2\phi}{dz^2}dz = \int_0^D \left[\frac{d}{dz}\left(\phi\frac{d\phi}{dz}\right) - \left(\frac{d\phi}{dz}\right)^2\right]dz$$

$$= \phi\frac{d\phi}{dz}\Big|_0^D - \int_0^D \left(\frac{d\phi}{dz}\right)^2 dz = -\int_0^D \left(\frac{d\phi}{dz}\right)^2 dz.$$

Therefore, the eigenvalue is

$$\lambda^2 = \frac{\int_0^D \left(\frac{d\phi}{dz}\right)^2 dz}{\int_0^D \frac{N^2}{f_o^{\ 2}}\phi^2 dz} \succ 0$$

In the case of a uniform *N*, the eigenfunctions and eigenvalues can be easily solved as

$$\phi_m(z) = \cos\left(\frac{N}{f_o}\lambda_m z\right) \tag{5.3.10}$$

$$\frac{N}{f_o}\lambda_m = \frac{m\pi}{D}, \qquad m = 0,1,2,\cdots \tag{5.3.11}$$

Substitute them into (5.3.8), we have the dispersion relationship

$$\omega = \frac{-\beta k}{k^2 + l^2 + \frac{f_0^2}{N^2}(\frac{m\pi}{D})^2}$$

Thus, for each vertical mode, *m*, the dispersion relationship is exactly the same as that for the shallow water Rossby wave, provided that we replace the effective deformation radius for each mode as:

$$L_{Dm}^2 = \left(\frac{ND}{f_o m\pi}\right)^2 \quad . \qquad (5.3.12)$$

The deformation radius vanishes, with an increasing *m*.

The correspondence of the deformation radius in (5.3.12) with that in the 1.5-layer model (1.5.4) can be readily seen below. Since

$$N^2 = \frac{g}{\rho_m}\frac{d\rho_s}{dz} \sim g\frac{\Delta\rho}{\rho_m}\frac{1}{D} = g'\frac{1}{D},$$

we have

$$L_{Dm}^2 = \frac{g'}{D}\frac{D^2}{f_o^2(m\pi)^2} = \frac{g'D}{f_o^2}\frac{1}{(m\pi)^2} = \frac{g'D_m}{f_o^2}$$

where

$$D_m = \frac{D}{(m\pi)^2}$$

is the equivalent depth. Thus, each baroclinic mode propagate exactly as a 1.5 layer model Rossby wave with an equivalent depth of D_m. (Some people also use the expression of $L_{Dm} = \frac{g\hat{D}_m}{f_o^2}$, such that the equivalent depth is $\hat{D}_m = \frac{\Delta\rho}{\rho}D_m$)

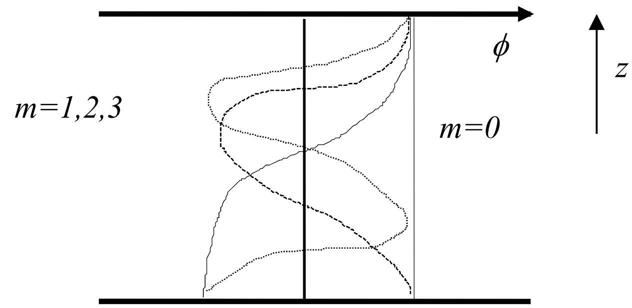
The L_{Dm} is called the internal (baroclinic) deformation radius. This gives a close analogy between shallow water dynamics and the stratified dynamics. In the ocean,

$$\frac{L_{D1}}{L_{Do}} = \frac{\Delta\rho}{\rho} << 1.$$

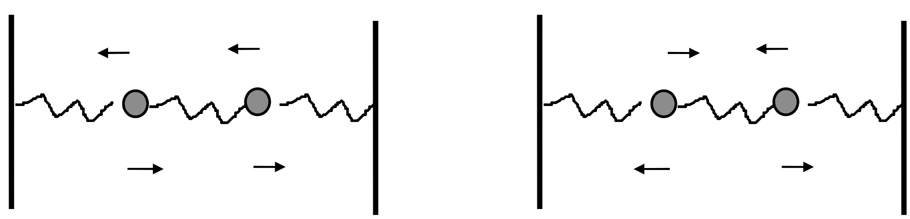
The vertical structure of the normal modes are further discussed below. The *m=0* mode is the barotropic mode or external mode (deformation radius infinitely large in the absence of free surface elevation here). The velocity does not have shear in the vertical direction and there is no density perturbation for this mode.

The *m ≥ 1* mode is the *mth* baroclinic mode or internal mode. These modes have *m* node points in the velocity field and are all accompanied with density perturbations. For all the baroclinic modes, the vertically integrated net transport vanishes.

This can be shown directly from (5.3.7) and (5.3.9). Integrate (5.3.7) from $z=0$ to D gives $\lambda_m^2 \int_0^D \phi_m(z')dz' = 0$ and therefore $\int_0^D \phi_m(z')dz' = 0$, if $\lambda_m \neq 0$.



Note 2: Equivalent particle examples of external and internal modes. Consider two balls connected by a spring. There are two possible normal modes. The first has both balls moving in the same direction, as if there is only one ball. This is the"barotropic mode". The second has the two balls always moving in the opposite directions. This is the "baroclinic mode".

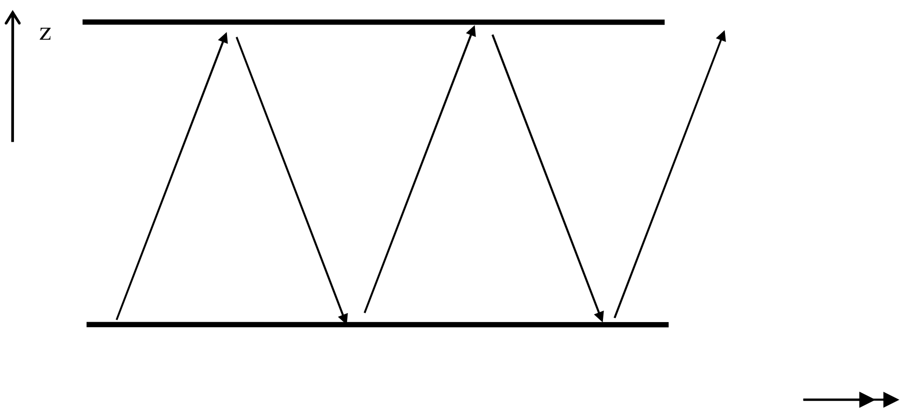


When more balls are added, there are more freedoms and more "modes". ||

In reality, *N(z)* is not uniform at all (Fig.5.3). Analytical solution becomes usually impossible. Nevertheless, for slowly varying *N(z),* we can still use the WKB method such that, with $U=0$, (5.2.5) becomes :

$$m^2 = -\frac{N^2(z)}{f^2}\left(\frac{\beta}{C} + k^2 + l^2\right)$$

for the oceanic case. If m is real at any height, it is real at all height, although N(z) may change. So there is no internal reflection. The normal mode is caused by reflection at the top and bottom boundaries.



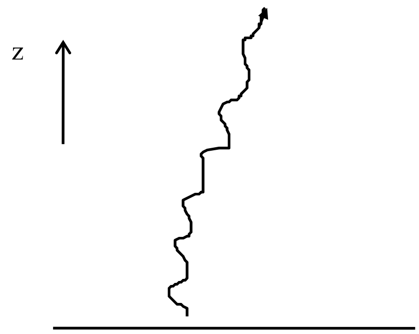
After a couple of reflections, normal mode is established in the z direction. The establishment of the normal mode is similar to the normal mode in the case of horizontal boundaries. The key is that the wave energy is trapped within a finite region.
The most dramatic change of N is in the oceanic thermocline. The WKB solution shows that $m(z)$ changes, small for small N, but large for large N. (Fig.5.4).

Indeed, for a general $N(z)$, the veritical eigenvalue problem (5.3.7) and (5.3.9) can be solved numerically to give λ_m. Then, the horizontal structure satisfies (5.3.8), which is the same as the shallow water Rossby wave, except for replacing the deformation radius by $L_D^2=\lambda_m^{-2}$. The Rossby wave of the mth mode has the dispersion relationship

$$\omega = \frac{-\beta k}{k^2 + l^2 + \lambda_m^{\ 2}}.$$

<u>2. Atmospheric Case</u>

The normal mode in the atmosphere is much more complicated, because of the lack of a upper boundary. It turns out that the normal mode usually doesn't exist in the atmosphere. This is not hard to imagine, because, in the absence of a reflective top boundary, a normal mode can't be established.



Mathematically, one can see this crudely here. If the vertical mode equation allows the solution

$$\phi \sim Ae^{imz} + Be^{-imz}$$

The absence of an energy source from above requires that the wave energy radiates upward only. This selects $B=0$. Furthermore, at the bottom, $\partial_z \phi = 0$ determines that $A = 0$. Therefore, there is no normal mode. However, normal modes may exist when a strong shear of U(z) produces internal reflections.

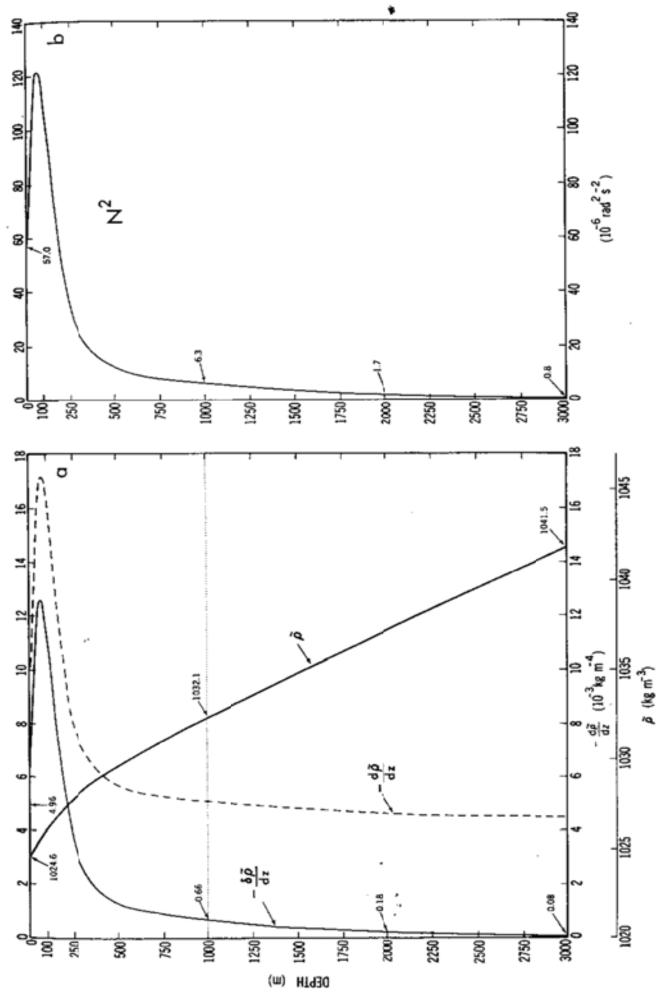


FIGURE 8.16. Vertical profiles of the global-mean density in kg m^{-3} and two measures of the mean static stability (see the text) in 10^{-3} kg m^{-4}(a), and a vertical profile of the global mean Brunt–Väisälä frequency squared (b) in 10^{-6} rad^2 s^{-2}.

Fig.5.3: Vertical profiles of N in the atmosphere and ocean

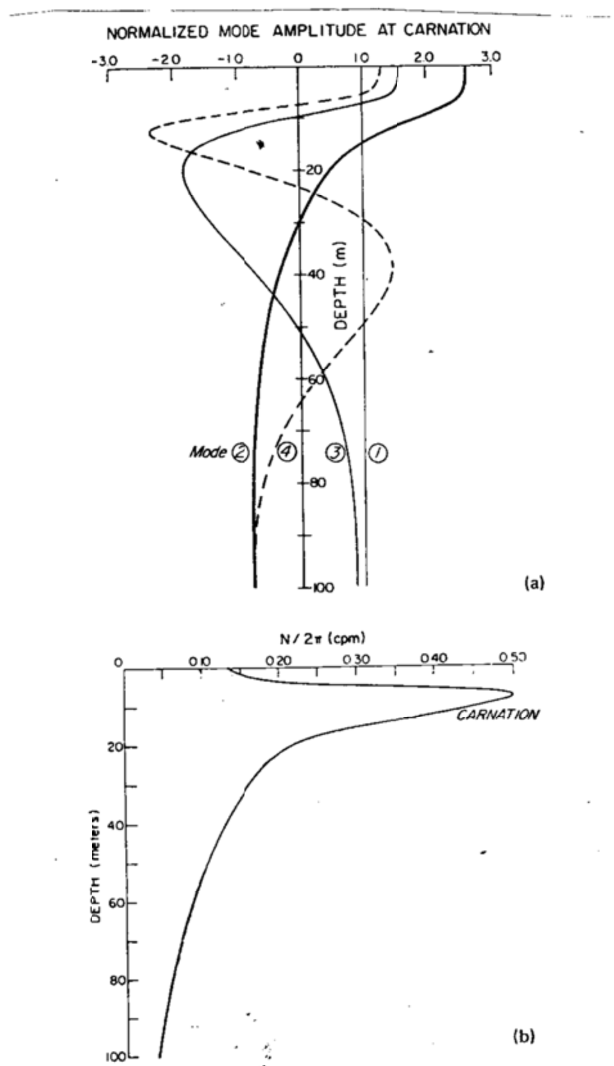


Figure 6.12.1 (a) The barotropic and first three baroclinic modes, as calculated by Kundu, Allen, and Smith (1975) for (b) the distribution of N observed at ocean station Carnation, off the Oregon coast.

Fig.5.4: Normal modes in the presence of realistic oceanic stratification.

Sec 5.4: The Eliassem-Palm Theorem

1. E-P theorem:

The shallow water E-P theorem in section 2.6 can be generalized to the stratified fluid. The QGPV equation.of the atmosphere is

$$D_g q = (\partial_t + u\partial_x + v\partial_y)q = S_q$$

Consider a basic state

$$U = U(y,z)\,,\ \Theta = \Theta(z)\,,\ \Psi = \Psi(y,z)\,,\ \phi = f_o\Psi(y,z) + \Phi(z)$$

where

$$\frac{\partial\Psi}{\partial y} = -U(y,z),\ \frac{d\Phi}{dz} = \frac{g\overline{\mathrm{T}}(z)}{T_s} = \frac{g}{T_s}(\frac{p}{p_*})^k\,\Theta$$

and the mean temperature field

$$T = T(y,z) + \overline{\mathrm{T}}(z) = T(y,z) + (\frac{p}{p_*})^k\,\Theta(z)$$

where the basic state satisfies the thermal wind relationship

$$-\frac{g}{T_s}\frac{\partial T}{\partial y} = f_o\frac{\partial U}{\partial z}$$

The mean QGPV is therefore:

$$Q(y,z) = f_o + \beta y + \frac{\partial^2\Psi}{\partial y} + \frac{1}{p}\frac{\partial}{\partial z}(\frac{pf_o^{\ 2}}{N^2}\frac{\partial\Psi}{\partial z})$$

and the mean PV gradient is

$$Q_y(y,z) = \beta - \frac{\partial^2 U}{\partial y^2} - \frac{1}{p}\frac{\partial}{\partial z}(\frac{pf_o^{\ 2}}{N^2}\frac{\partial U}{\partial z})$$

Write

$$\psi = \Psi(y,z) + \psi'(x,y,z,t)$$

where $\psi' << \Psi$, is a small perturbation. The QGPV equation can be linearized as

$$(\partial_t + U\partial_x)q' + v'Q_y = S'_q \qquad (5.4.1)$$

Multiplying the equation by pq'/Q_y, we have

$$(\partial_t + U\partial_x)\frac{pq'^2}{2Q_y} + pv'q' = pq'S'_q/Q_y \qquad (5.4.2)$$

Since

$$v' = \partial_x \psi', \; q' = \partial_{xx}\psi' + \partial_{yy}\psi' + \frac{1}{p}\partial_z \left(p \frac{f_o}{N^2}\partial_z \psi'\right)$$

$$v'\partial_{xx}\psi' = \frac{1}{2}\partial_x[(\partial_x\psi')^2]$$

$$v'\partial_{yy}\psi' = \partial_y[\partial_x\psi'\partial_y\psi'] - \frac{1}{2}\partial_x[(\partial_y\psi')^2]$$

$$v'\frac{\partial}{\partial z}[p\frac{f_o^{\,2}}{N^2}\partial_z\psi'] = \partial_z[p\frac{f_o^{\,2}}{N^2}\partial_x\psi'\partial_z\psi'] - \partial_{xz}^2\psi'\frac{pf_o^{\,2}}{N^2}\partial_z\psi'$$

$$= \partial_z[p\frac{f_o^{\,2}}{N^2}\partial_x\psi'\partial_z\psi'] - \frac{1}{2}\partial_x[\frac{pf_o^{\,2}}{N^2}(\partial_z\psi')^2]$$

pv'q' can be written in the form of flux divergence

$$pv'q' = \nabla \cdot \mathbf{F} \qquad (5.4.3)$$

Here, the flux **F** is the generalized E-P flux

$$\mathbf{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \frac{p}{2}[(\psi_x')^2 - (\psi_y')^2 - \frac{f_o^{\,2}}{N^2}(\partial_z\psi')^2] \\ p\partial_x\psi'\partial_y\psi' \\ \frac{pf_o^{\,2}}{N^2}\partial_x\psi'\partial_z\psi' \end{bmatrix} = \begin{bmatrix} \frac{p}{2}[v'^2 - u'^2 - (N\frac{\theta'}{d\Theta/dz})^2] \\ -pu'v' \\ \frac{pf_o v'\theta'}{d\Theta/dz} \end{bmatrix} \qquad (5.4.4)$$

and

$$N^2 = \frac{g}{\Theta_s}\frac{d\Theta}{dz}, \quad \partial_z\psi' = \frac{\theta'}{f_o}\frac{N^2}{d\Theta/dz}$$

The perturbation PV equation (5.4.2) can be written in the wave activity equation:

$$(\partial_t + U\partial_x)A + \nabla\cdot\mathbf{F} = S \qquad (5.4.5)$$

where

$$A = \frac{pq'^2}{2Q_y} \qquad (5.4.6)$$

is the wave activity, and $S = S_q pq'/Q_y$. The conventional E-P equation is the zonal mean of the generalized E-P equation.

$$\partial_t\overline{A} + \nabla\cdot\overline{\mathbf{F}} = \overline{S} \qquad (5.4.7)$$

where

$$\overline{A} = \frac{p\frac{1}{2}\overline{q'^2}}{Q_y}, \quad \overline{\mathbf{F}} = \begin{bmatrix} \overline{F}_y \\ \overline{F}_z \end{bmatrix} = \begin{bmatrix} -p\overline{u'v'} \\ pf_o \dfrac{\overline{v'\theta'}}{d\Theta/dz} \end{bmatrix} \tag{5.4.8}$$

This gives the Elliassn - Palm Theorem:

For (i) steady amplitude $\partial_t A = 0$, and (ii) conservative $S = 0$, the E-P flux **F** is non-divergent. (Therefore, **F** can't originate from nowhere and end in nowhere, like the mass flux of an incompressible fluid)

Note 1: For the ocean:

$$\mathbf{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} (\psi'_x)^2 - (\psi'_y)^2 - \dfrac{f_o^2}{N^2}(\partial_z \psi')^2 \\ \partial_x \psi' \partial_y \psi' \\ \dfrac{f_o^2}{N^2}\partial_x \psi' \partial_z \psi' \end{bmatrix} = \begin{bmatrix} u'^2 - v'^2 - (N\dfrac{\rho}{d\rho_S/dz})^2 \\ -u'v' \\ \dfrac{f_o v'\rho}{d\rho_s/dz} \end{bmatrix} |||$$

2. Wave Activity Flux and Group Velocity

For almost-plane waves, under the WKB assumption, the solution can be assumed of the form

$$\psi'(x,y,z,t) = \mathrm{Re}\left[\Psi(y,z)e^{i\theta + \frac{z}{2H}}\right]$$

where $\theta = k(x-ct) + \int^y l(y')dy' + \int^z m(z')dz'$. We can derive the wave activity as

$$A = \frac{p}{2}\overline{q'^2}/Q_y = \frac{p}{4\Delta}|\Psi|^2$$

where we have used

$$q' \approx \left[k^2 + l^2 + \frac{f^2}{N^2}\left(m^2 + \frac{1}{4H^2}\right)\right]\psi'.$$

and (5.2.7) with

$$\Delta = Q_y[k^2 + l^2 + \frac{f_o^2}{N^2}(m^2 + \frac{1}{4H^2})]^{-2} = \frac{(U-c)^2}{Q_y}$$

Similarly, the flux is

$$\overline{F}_y = p\overline{\partial_x \psi' \partial_y \psi'} \approx \frac{1}{2} kl|\Psi|^2$$
$$\overline{F}_z = p\frac{f_0^{\,2}}{N^2}\overline{\partial_x \psi' \partial_z \psi'} \approx \frac{1}{2}\frac{f_0^{\,2}}{N^2}\mathrm{Re}\left[ik\Psi\left[\left(im + \frac{1}{2H}\right)\Psi\right]^*\right] \approx \frac{1}{2}\frac{f_0^{\,2}}{N^2} km|\Psi|^2 \tag{5.4.9}$$

Notice the group velocity of the baroclinic Rossby waves in (5.2.6), we therefore have

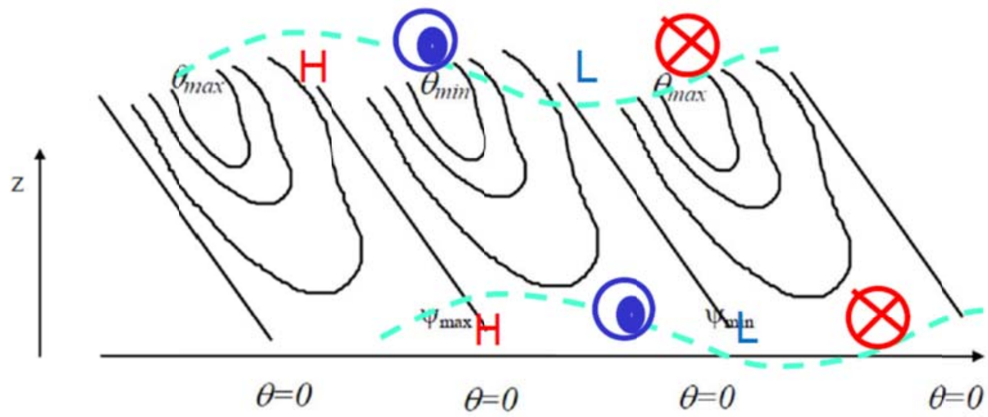
$$\overline{\mathbf{F}} = \left(\overline{F}_y, \overline{F}_z\right) = \left[2kl\Delta, \quad 2\frac{f_0^{\,2}}{N^2} km\Delta\right] A = (C_{gy}, C_{gz})A\,.$$

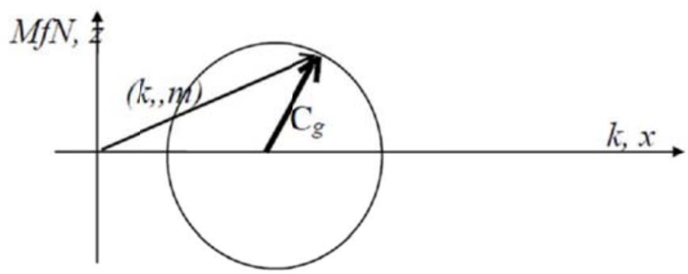
3 Vertical Propagation and Meridianal Heat Transport

The vertical component of the E-P flux is directly related to the meridional heat flux

$$F_z = \frac{f_0^{\,2}}{N^2}\overline{\partial_x \psi' \partial_z \psi'} = \frac{\theta'}{f_o}\frac{N^2}{d\Theta/dz}\overline{v'\theta'} \quad \propto \quad \overline{v'\theta'} \quad \propto km\,|\,\Psi\,|^2 \tag{5.4.10}$$

An upward E-P flux ($F_z > 0$) corresponds to a northward heat transport, and vise versa. In the atmosphere, Rossby waves are usually forced from the surface to propagate upward. ($F_z > 0$). This corresponds to a westward tilt (*km*>0) and should transport heat poleward. (Now, the geopotential perturbation is 90o phase lead of θ' and therefore v' is in phase with θ'). In addition, waves are also caused by baroclinic instability (Chapter 6). The unstable waves also tilt westward and transport heat poleward.

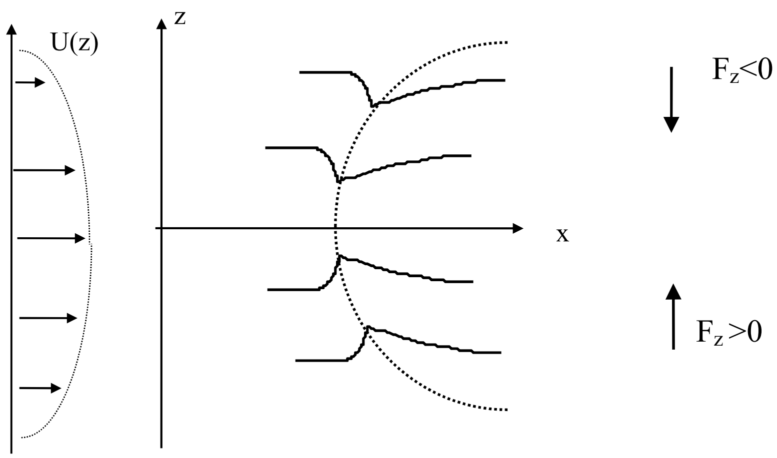




The northward heat transport by Rossby waves contributes to the major part of the atmospheric poleward heat transport in the midlatitude region. Interestingly, in the ocean, the wind forces Rossby waves at the surface and therefore downward. These waves will transport heat equatorward, against the mean gradient.

<u>4. Applications</u>

Case 1: Wave-mean flow interaction in vertically sheared flow. Assuming a westerly wind with a maximum speed in the middle level. In the lower half, the westward tilting trough produces an upward E-P flux. The accompanied northward heat transport is down the mean temperature gradient ($T_y<0$ for $U_z>0$) and therefore tends to reduce the mean temperature gradient. The perturbation grows by extracting APE from the mean APE. Similar discussions show that the perturbation in the upper half is also unstable. Alternatively, the E-P flux converges, increasing the wave activity at the expense of the mean flow strength. (∂_tA increases and ∂_tU decreases).



Question: Indeed, from the QG thermodynamic equation

$$\partial_t \theta + \nabla(\mathbf{u}_g \theta) + w\Theta_z = Q$$

Zonal mean we have

$$\partial_t \overline{\theta} + \partial_y \overline{(v_g{}'\theta')} + \overline{w}\Theta_z = \overline{Q}$$

That is

$$\partial_t \overline{\theta} = -\partial_y \overline{(v_g{}'\theta')} - \overline{w}\Theta_z + \overline{Q}.$$

how to get vertical gradient of Fz impact on mean temperature?

<u>Case 2</u>:

Vertical propagation of the atmospheric Rossby waves (see the end of last section). The amplitude increases for a vertically propagation wave with height inversely proportional to pressure. For a wave packet originate at the surface (1000mb) propagating into the stratosphere (10mb), its amplitude increases by 10 times. This can be seen using the E-P theorem. For steady, conservation waves,

$$\nabla \bullet \mathbf{F} = 0.$$

For plane waves, F_y is independent of y, so that

$$\partial_z \overline{F} = -\partial_y \overline{F} = 0,$$

or

$$\partial_z\left[\frac{pf_o^2}{N^2}\overline{\psi'_x\psi'_z}\right]=0 \qquad (5.4.11)$$

If N is constant,

$$\partial_z\left[p\overline{\psi'_x\psi'_z}\right]=0 \qquad (5.4.12)$$

Now, if

$$\psi'=\mathrm{Re}[\Gamma(y,z)e^{i\theta}]$$

where $\Gamma=\Psi e^{\frac{z}{2H}}$. We have $\partial_x\psi'=\mathrm{Re}(ik\Gamma e^{i\theta})$, $\partial_z\psi'=\mathrm{Re}[(im+\frac{1}{2H})\Gamma e^{i\theta}]$. Therefore,

$\overline{\partial_x\psi'\partial_z\psi'}=\frac{1}{2}km\,|\,\Gamma\,|^2$. Thus, (5.4.12) gives

$$\partial_z(p\,|\,\Gamma\,|^2)=0$$

In reality, the amplitude can be changed by dissipation, nonlinearity, wave refraction , etc

Questions for Chapter 5

Q5.1. (buoyancy and vorticity forced responses) A fluid is bounded by flat top and bottom at z=0 and -H, and is forced by external momentum forcing ***F*** and buoyancy forcing Q. The linearized QGPV equation is

$$(\partial_t + U_0 \partial_x) q' + v' \overline{q}_y = \frac{1}{\rho_m} curl(\mathbf{F}) - \partial_z \left(\frac{g f_o Q}{\rho_m N^2} \right)$$

where $q' = \partial_{xx}\psi + \partial_{yy}\psi + \partial_z \left[\frac{f_o^2}{N^2} \frac{\partial \psi}{\partial z} \right]$, $v' = \partial_x \psi$, and the mean flow U_0 is assumed to be independent of height and therefore the mean PV gradient is $\overline{q}_y = \beta$.

(a) A momentum forcing ***F*** has a vertically integrated vorticity component such that $\int_{-H}^{0} curl\mathbf{F} dz \neq 0$ (e.g. a surface wind curl forcing or bottom drag curl) . Show that this forcing can generate barotropic response $\int_{-H}^{0} \psi dz \neq 0$. (hint: vertically integrate the PV equation).

(b) An internal buoyancy forcing Q vanishes on the top and bottom boundaries $Q(x, y, z, t) = 0$ at z=0 and -H (such as an internal heating in the middle of the fluid). Show that the forced response has no barotropic component $\int_{-H}^{0} \psi dz = 0$.

What happens if the buoyancy forcing or density perturbation do not vanish on the top and bottom boundaries. Why?

(c) What could happen if the mean flow varies with height?

Exercises for Chapter 5

E5.1. (Rossby height) The basic state is motionless and has a constant Brunt-Vasara frequency *N*. Assume a boundary perturbation of geostrophic streamfunction at the level z_1 of the form $\psi(x,y,z=0,t)=Ae^{i(kx-\omega t)}$. If this perturbation does not generate PV anomaly away from the disturbance level ($z>z_1$ *and* $z<z_1$), that is the PV satisfies the equation

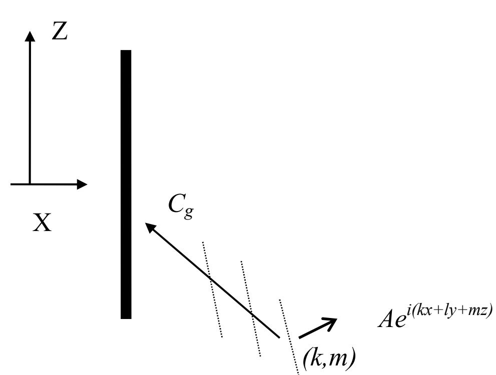
$$q=\partial_{xx}\psi+\partial_{yy}\psi+\frac{f_o^2}{N^2}\partial_{zz}\psi=0\,,$$

away from the forcing, how deep can the boundary perturbation penetrate?

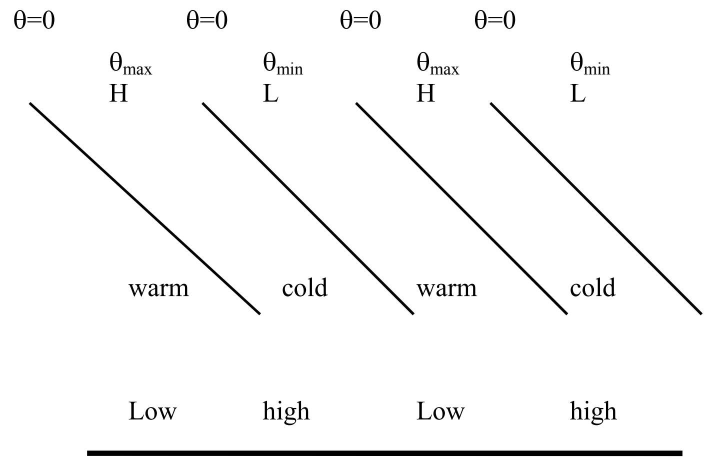
E5.2. (Vertical Rossby wave propagation diagram) The basic state is motionless and has a constant Brunt-Vasara frequency *N*. Baroclinic Rossby waves have the form of $e^{i(kx+ly+mz)}$

a) Discuss the mathematical similarity between the vertical propagation of stratified baroclinic Rossby waves and the meridional propagation of shallow water Rossby waves. Plot the wave vector and direction of the group velocity in the wave number *(k,m)* plane (or the dispersion diagram circle).

b) In light of (a), consider an upward/westward propagating baroclinic Rossby wave that is incident on a vertical wall (or tall mountain). What will be the direction of the reflected Rossby wave ? What will be the wave phase pattern?



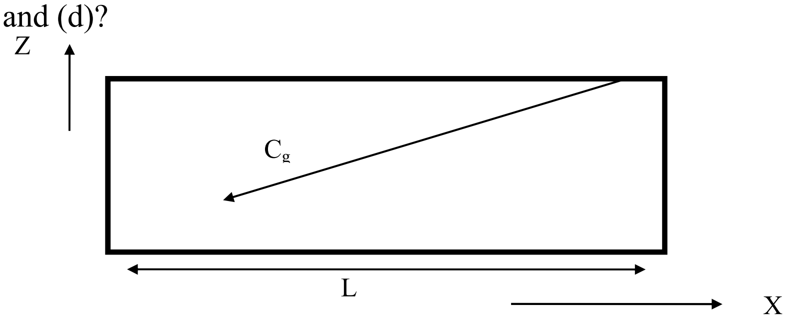
(c) If a baroclinic system tilts westward with height, what is the direction of Rossby wave energy propagation?



What kind of weather synoptic system does this correspond to? Which direction does this system transport heat flux? What is the direction of the E-P flux in the vertical direction? (For simplicity, you can assume an infinite scale height, or incompressibility).

E5.3. (Wind forced stratified ocean) A stratified linear ocean is forced by a spatially uniform Ekman pumping on the surface with a frequency σ. The ocean basin has a zonal scale L (such that the wave number is $k=2\pi/L$) that is much longer than the internal deformation radius:

(a) Find the direction of the downward group velocity.
(b) What is the direction of the group velocity when the forcing frequency approaches zero (the limit of steady forcing)?
(c) In light of (b), is it possible to have subsurface motion under a steady wind forcing?
(d) Is Sverdrup relation valid in the limit of a steady wind?
(e) What is the implication of (c) and (d)?



E5.4 (Non-Doppler shift effect) In a stratified ocean, we will consider planetary scale perturbations that are governed by the potential vorticity equation

$$\partial_t\left[\partial_z\left(\frac{f_o^{\ 2}}{N^2}\frac{\partial\psi}{\partial z}\right)\right]+\beta\partial_x\psi+J\left[\psi,\partial_z\left(\frac{f_o^{\ 2}}{N^2}\frac{\partial\psi}{\partial z}\right)\right]=0 \quad .$$

We project the streamfunction on vertical modes: $\psi=\sum_{m=0}^{\infty}\Psi_m(x,y,t)\phi_m(z)$ where the *mth* vertical mode ϕ_m is determined by the eigenvalue equation (5.3.7) as

$$\frac{d}{dz}\left[\frac{f_o^{\ 2}}{N(z)^2}\frac{d\phi_m(z)}{dz}\right]=-\lambda_m^{\ 2}\phi_m(z).$$

(a) If the flow is projected only on a single vertical mode *m=M*, what is the advection term in the potential vorticity equation.

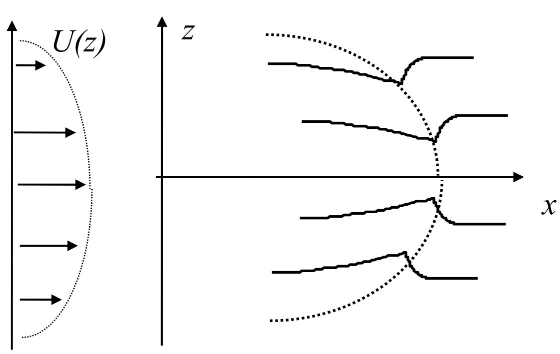
(b) In light of (a), how do you interpret the perfect non-Doppler-shift effect of the planetary Rossby wave in the shallow water or the 1.5-layer model?

(c) If the flow is projected on more than one vertical modes, show that the *mth* mode of the flow only advects the part of the stretching voriticty that excludes mode *m*.

(d) Based on (c), under what condition, Rossby waves will be advected by mean flow (or Doppler-shift occurs) in a general continuously stratified ocean?

E5.5 (Normal modes in the presence of topography) In the presence of a north-south bottom topography $z_B=\Lambda y$, derive the normal modes of Rossby waves in section 5.3 in the case of a constant Brunt-Vasara frequency. (Hint: now the bottom boundary condition at z=0 is $w(x,y,z_B)=\vec{u}_B\bullet\nabla z_B=\Lambda\partial_x\psi(x,y,0)$).

E5.6: (Wave-mean flow interaction of baroclinic waves). Based on the wave activity equation and E-P flux (5.4.7) and (5.4.8), discuss the wave-mean flow interaction of the following disturbances in a westerly shear flow.



(a) Will the disturbance grow or decay? Will the mean flow intensify or weaken?
(b) Discuss the difference and similarity from the corresponding barotropic case (in section 2.6) of negative viscosity.