Ch4: Basics of Stratified Fluid

Sec. 4.1: Basic Equations

Stratification will introduce new physics. We first derive the equations. Denoting rotation vector as $\mathbf{\Omega}$, gravity potential Φ, $\mathbf{u}=(u,v,w)$, $\nabla=\mathbf{i}\partial_x+\mathbf{j}\partial_y+\mathbf{k}\partial_z$. The momentum equations are

$$\rho\left(\frac{d\mathbf{u}}{dt}+2\mathbf{\Omega}\times\mathbf{u}\right)=-\nabla p-\rho\nabla\Phi+\mathbf{F}. \qquad (4.1.1)$$

The mass equation is

$$\frac{d\rho}{dt}+\rho\nabla\bullet\mathbf{u}=0 \qquad \text{or} \qquad \frac{\partial\rho}{\partial t}+\nabla\bullet\rho\mathbf{u}=0 \qquad (4.1.2)$$

The equation of state in general is

$$\rho=\rho(p,T,S...) \qquad (4.1.3)$$

In the ocean, neglecting salinity, the equation of state is

$$\rho=\rho_m\left[1-\alpha(T-T_0)\right] \qquad (4.1.4)$$

where $\alpha=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_P$ is the coefficient of thermal expansion and ρ_m is a constant reference density, which can be chosen as the average density.

In the atmosphere, using the perfect gas, the equation of state is

$$\rho=\frac{P}{RT} \qquad (4.1.5)$$

where R is the gas constant.

The thermodynamic equation describes the internal energy change. For the ocean, which is incompressible, the thermodynamic equation is

$$\rho C_p\frac{dT}{dt}=\rho Q+\tilde{k}\nabla^2T \qquad (4.1.6)$$

where $\tilde{k}$ is the thermal conductivity, Q is the heating rate per unit mass. This can be rewritten as

$$\frac{dT}{dt}=J+k\nabla^2T \qquad (4.1.6a)$$

where $J=\frac{Q}{C_p}$ and $k=\frac{\tilde{k}}{\rho C_p}$.

The atmosphere is compressible, so the thermodynamic equation is

$$\frac{dT}{dt}-\frac{1}{\rho C_p}\frac{dp}{dt}=J+k\nabla^2 T \tag{4.1.7}$$

Define the potential temperature as $\theta = T\left(\frac{p_0}{p}\right)^{\gamma}$, where $\gamma = R/C_p$, we have

$$\ln\theta = \ln T + \ln\left(\frac{p_0}{p}\right)^{\gamma} = \ln T - \gamma \ln p + const$$

$$\frac{d\theta}{\theta}=\frac{dT}{T}-\gamma\frac{dp}{p}$$

$$d\theta = \left(\frac{\theta}{T}\right)\left(dT-\frac{\gamma T}{p}dp\right)=\left(\frac{p_0}{p}\right)^{\gamma}\left(dT-\frac{\gamma T}{p}dp\right)$$

$$\frac{d\theta}{dt}=\left(\frac{p_0}{p}\right)^{\gamma}\left\{J+k\nabla^2 T\right\} \tag{4.1.8}$$

where we have used (4.1.7) and the ideal gas law (4.1.5), such that $\frac{\gamma T}{p}=\frac{RT}{pC_p}=\frac{1}{\rho C_p}$.

Therefore, in the local Cartesian coordinate, we have the full set of equations for the stratified ocean and atmosphere as:

$$\partial_t u + u\partial_x u + v\partial_y u + w\partial_z u - fv = -\frac{1}{\rho}\partial_x p + \frac{1}{\rho}F_x$$

$$\partial_t v + u\partial_x v + v\partial_y v + w\partial_z v + fu = -\frac{1}{\rho}\partial_y p + \frac{1}{\rho}F_y \tag{4.1.8}$$

$$\partial_t w + u\partial_x w + v\partial_y w + w\partial_z w + g = -\frac{1}{\rho}\partial_z p + \frac{1}{\rho}F_z$$

$$\partial_t \rho + u\partial_x \rho + v\partial_y \rho + w\partial_z \rho = -\rho\left\{\partial_x u + \partial_y v + \partial_z w\right\} \tag{4.1.9}$$

$$\partial_t T + u\partial_x T + v\partial_y T + w\partial_z T = J + k\nabla^2 T \quad oceam \tag{4.1.10a}$$

$$\partial_t \theta + u\partial_x \theta + v\partial_y \theta + w\partial_z \theta = \left(\frac{p_0}{p}\right)^{\gamma}\left\{J+k\nabla^2 T\right\} \quad atmosphere \tag{4.1.10b}$$

$$\rho \approx \rho_m\left[1-\alpha\left(T-T_0\right)\right] \quad ocean \tag{4.1.11a}$$

$$\rho = \frac{P}{RT} \quad atmosphere \tag{4.1.11b}$$

where a beta-plane is used such that $f = f_0 + \beta y$. The characteristics of the ocean and atmosphere also allow the equations to be further simplified for each system.

1. Oceanic Equations

The ocean is almost incompressible. Therefore, $\rho = \rho_m + \rho_0(x,y,z,t)$ with $\frac{\rho_0}{\rho_m} << 1$. The incompressibility has three consequences. First in the mass equation, $\frac{d\rho_0}{dt} \Big/ \rho \approx \frac{\rho_0}{\rho_m} << 1$, and therefore (4.1.2) reduces to volume conservation

$$\nabla \bullet \mathbf{u} = 0 \tag{4.1.12}$$

Second, the thermodynamic equation and the equation of state can be combined together. Eqn.(4.1.11a) can be written as $\rho_0 = -\rho_m \alpha T'$ where $T' = T - T_0$. The thermodynamic equation (4.1.10a) can therefore be written as

$$\partial_t \rho_0 + (\mathbf{u} \bullet \nabla)\rho_0 = k\nabla^2 \rho_0 - \alpha J \equiv S_0 \tag{4.1.13}$$

Third, large scale oceanic process also satisfy *D/L<<1* and in turn the hydrostatic approximation (as in the case of shallow water in section 1.1). The vertical momentum equation at the leading order can be shown as

$$\frac{\partial p}{\partial z} = -g\rho$$

Defining $p = P(z) + p_0(x,y,z,t)$, where $P = -g\rho_m z$ is the static pressure due to the average density, the rest of perturbation pressure p_0 satisfies

$$\frac{\partial p_0}{\partial z} = -g\rho_0 \tag{4.1.14}$$

Fourth, the momentum equations can be further simplified using the Boussinesq approximation, such that the density is a constant except when it represents the buoyancy forcing in the vertical momentum equation. The horizontal pressure gradient forcing can be approximated as

$$\frac{1}{\rho}\nabla_h p = \frac{1}{\rho}\nabla_h p_0 \approx \frac{1}{\rho_m}\nabla_h p_0$$

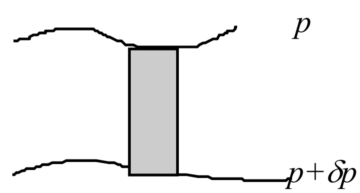
where $\nabla_h = \mathbf{i}\partial_x + \mathbf{j}\partial_y$ is the horizontal gradient.

Finally, after the four more approximations, we have the ocean equations as:

$$\partial_t u + (\mathbf{u} \bullet \nabla) u - fv = -\frac{1}{\rho_m} \partial_x p_0 + \frac{1}{\rho_m} F_x$$
$$\partial_t v + (\mathbf{u} \bullet \nabla) v + fu = -\frac{1}{\rho_m} \partial_y p_0 + \frac{1}{\rho_m} F_y$$
$$\frac{1}{\rho_m} \partial_z p_0 = -\frac{g\rho_0}{\rho_m} \tag{4.1.15}$$
$$\partial_x u + \partial_y v + \partial_z w = 0$$
$$\partial_t \rho_0 + (\vec{\mathbf{u}} \bullet \nabla) \rho_0 = S_0$$

2. Atmospheric Equations

Unlike the ocean, the atmosphere is very compressible. The mass equation (4.1.9) therefore has to remain a predictive equation in general. This equation, however, can be simplified for large scale *D/L<<1* processes by using the hydrostatic approximation $\partial_z p = -g\rho$. The mass per unit area contained between the pressure surface p and $p+\delta p$ is



$$\rho\, \delta z = \frac{\delta p}{g}$$

The mass of a material element is:

$$\delta m = \rho \delta x\, \delta y\, \delta z = \frac{\delta x \delta y\, \delta p}{g}$$

Following the flow, the mass conservation $\frac{d}{dt}(\delta m) = 0$ becomes

$$\frac{d}{dt}(\delta m) = \frac{1}{g}\left\{ \delta y \delta p \frac{d}{dt}(\delta x) + \delta x \delta p \frac{d}{dt}(\delta y) + \delta x \delta y \frac{d}{dt}(\delta p) \right\} = 0$$
$$= \frac{\delta x \delta y \delta p}{g} \left\{ \frac{1}{\delta x} \delta\left(\frac{dx}{dt}\right) + \frac{1}{\delta y} \delta\left(\frac{dy}{dt}\right) + \frac{1}{\delta p} \delta\left(\frac{dp}{dt}\right) + \right\}$$
$$= \frac{\delta x \delta y \delta p}{g} \left(\partial_x u + \partial_y v + \partial_p \omega = 0 \right)$$

where $\omega = \frac{dp}{dt}$. The continuity equation for the atmosphere is therefore

$$\partial_x u + \partial_y v + \partial_p \omega = 0 \tag{4.1.16}$$

This is a great simplification due to the *p*-coordinate. But, for other purposes, it turns out not to be very convenient to use the *p*-coordinate. We can use the log *p* which combines both the advantages of the *p* and *z* coordinates.

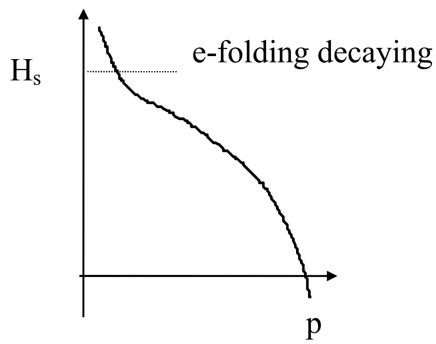
Since $p = \rho RT$, we have

$$\frac{\partial p}{\partial z} = -g\rho = -\frac{g}{RT}p$$

$$p = p_o e^{-\int_0^z \frac{g}{RT(z')}dz'} = p_o e^{-\int_0^z \frac{dz'}{H_s(z')}}$$

Since T is roughly constant (in Kelvin) with height, say $T \approx T_s$, p decreases roughly exponentially with a scale height

$$p = p_o e^{-\frac{z}{H_s}},$$



with a scale height

$$H_s = \frac{RT_s}{g}. \qquad (4.1.17)$$

(the ocean can be taken as the case of an infinite scale height). So wave-like motions, which have a simple form in *z*, have a more cumbersome mathematical structure in *p*. To avoid the problem, we use the height-like vertical coordinate:

$$Z = -H_s \ln\left(\frac{p}{p_o}\right) \qquad (4.1.18)$$

or

$$p = p_o e^{-\frac{Z}{H_s}}, \qquad (4.1.18a)$$

where *H* is constant (so *Z* is still in *p*-coordinate). We could in principle choose *H* to be anything, but *Z* will be like the real height if we choose *H* to be a typical value of the scale height H_s for the region of interest. For example, if we choose $T_s = 250$ K, then $H_s = \frac{R}{g}T_s = 7.3Km$.

For US standard atmosphere ($P_o = 1013.25hPa$)

p(hPa)	z(km)	$\bar{z}$ (km)
1000	0.111	0.096
850	1.457	1.282
700	3.012	2.700
500	5.534	5.156
300	9.164	8.885
200	11.784	11.884
100	16.18	16.904
50	20.576	21.965
30	23.849	25.694
10	31.055	33.714

Now, we will write Z_g for the real geometric height (w_g for vertical velocity in Z_g).

$$\omega = \frac{dp}{dt}$$

$$Z = -H \ln p + H \ln p_o$$

$$dZ = -H \frac{dp}{p}$$

$$w = \frac{dZ}{dt} = -\frac{H}{p}\frac{dp}{dt} = -\frac{H}{p}\omega$$

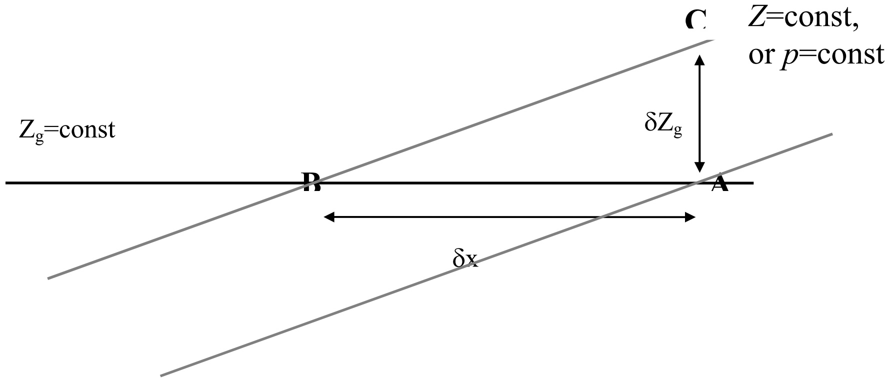
$$\frac{d\omega}{dp} = -\frac{H}{p}\frac{d\omega}{dZ} = \frac{H}{p}\frac{d}{dZ}\left(\frac{pw}{H}\right) = \frac{1}{p}\frac{d}{dZ}(pw)$$

The continuity equation in the log p or Z-coordinate can be derived from (4.1.16) as.

$$\partial_x u + \partial_y v + \frac{1}{p}\partial_Z (pw) = 0 \qquad (4.1.19)$$

Furthermore, in the *p*- (or Z-) coordinate, we also have to change the form of the horizontal pressure gradient. Take the u-equation as an example.

$$\frac{d}{dt}u - fv = -\frac{1}{\rho}(\partial_x p)_{y,z_g} + \frac{1}{\rho}F_x$$



$$\left(\partial_x p\right)_{Z_g} = \frac{p_A - p_B}{\delta x} = \frac{p_A - p_C}{\delta x} = \frac{g\rho(Z_{gA} - Z_{gC})}{\delta x} = \frac{g\rho(Z_{gB} - Z_{gC})}{\delta x} = \rho\left(\frac{\partial \Phi}{\partial x}\right)_Z,$$

Here, we have used $p_B = p_C$, $Z_{gA} = Z_{gB}$, the hydrostatic approximation $\delta p = -g\rho\delta Z_g$, so that $p_A - p_C = -g\rho(Z_{gA} - Z_{gC})$ and the definition of geopotential height as $\Phi = gZ_g$. This also shows another advantage of the *p*- (or log *p*-) coordinate: it gets rid of the $\frac{1}{\rho}$ factor in the pressure gradient term and therefore acts similar to the Bousinessq approximation in the ocean. Similarly, we have

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_{Z_g} = \left(\frac{\partial \Phi}{\partial y}\right)_Z$$

Finally the hydrostatic approximation is

$$\frac{\partial p}{\partial Z_g} = -g\rho \Rightarrow \frac{\partial Z_g}{\partial p} = -\frac{1}{g\rho}$$

$$\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho}$$

$$\frac{\partial \Phi}{\partial Z} = \frac{p}{H\rho} = g\frac{T}{T_s}$$

The complete set of the atmospheric equations are therefore

$$\partial_t u + (\mathrm{u} \bullet \nabla)u - fv = -\partial_x \Phi + \frac{F_x}{\rho}$$

$$\partial_t v + (\mathrm{u} \bullet \nabla)v + fu = -\partial_y \Phi + \frac{F_y}{\rho}$$

$$\partial_x u + \partial_y v + \frac{1}{p}\partial_Z (pw) = 0 \qquad (4.1.20)$$

$$\partial_Z \Phi = g\frac{T}{T_s} = \frac{g}{T_s}\left(\frac{p}{p_o}\right)^{\gamma} \theta \equiv g\frac{\theta}{\theta_s}$$

$$\partial_t \theta + (\mathrm{u} \bullet \nabla)\theta = Q_a$$

where $\theta_s = T_s\left(\frac{p_o}{p}\right)^{\gamma}$, $p = p_o e^{-\frac{Z}{H_s}}$, and $Q_a = \left(\frac{p_o}{p}\right)^{\gamma}\left\{J + k\nabla^2 T\right\}$.

A comparison of (4.1.15) and (4.1.20) shows that the ocean equations (4.1.15) can be recovered from the atmospheric equations (4.1.20) by the substitution of

$$p_o \Leftrightarrow p, \quad \frac{p'}{\rho} \Leftrightarrow \Phi, \quad \frac{\rho'}{\rho_0} \Leftrightarrow -\frac{\theta}{\theta_s}, \quad S_0 \Leftrightarrow -\frac{\rho_0}{T_s}Q_a$$

and by interpreting Z as the geometric height in (4.1.20).

Sec 4.2: Vorticity Equation and Circulation Theorem

1.Vorticity Equation

The derivation of the vorticity equation in a stratified fluid is similar to that in the shallow water case but now with three dimensional components. In addition, stratification adds terms to the equations. The momentum equations are:

$$\partial_t \mathbf{u} + (\mathbf{u} \bullet \nabla)\mathbf{u} + 2\Omega \times \mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla\Phi + \frac{1}{\rho}\mathbf{F}.$$

We will use the vector operation. The absolute vorticity is

$$\zeta_a = 2\Omega + \nabla \times \mathbf{u} = 2\Omega + \zeta$$

Notice:

$$\mathbf{A} \times \mathbf{B} \equiv \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_X & A_y & A_Z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\mathbf{i} + (A_z B_x - A_y B_z)\mathbf{j} + (A_X B_y - A_y B_X)\mathbf{k}$$

Using the identity $\boldsymbol{\zeta} \times \mathbf{u} = (\nabla \times \mathbf{u}) \times \mathbf{u} = (\mathbf{u} \bullet \nabla)\mathbf{u} - \frac{1}{2}\nabla(\mathbf{u} \bullet \mathbf{u})$,

we have:

$$\partial_t \mathbf{u} + (2\Omega + \boldsymbol{\zeta}) \times \mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla(\Phi + \frac{\mathbf{u} \bullet \mathbf{u}}{2}) + \frac{\mathbf{F}}{\rho}$$

$\nabla \times (\mathbf{u} - eq.)$, we have

$$\partial_t \boldsymbol{\zeta} + \nabla \times \{\boldsymbol{\zeta}_a \times \mathbf{u}\} \equiv \frac{\partial \boldsymbol{\zeta}_a}{\partial t} + \nabla \times (\boldsymbol{\zeta}_a \times \mathbf{u}) = -\nabla \times (\frac{1}{\rho}\nabla P) + \nabla \times (\frac{\mathbf{F}}{\rho})$$

Since

$$(i) \quad \nabla \times \left\{\frac{1}{\rho}\nabla P\right\} = -\frac{1}{\rho^2}\nabla\rho \times \nabla P + \frac{1}{\rho}[\nabla \times \nabla P]$$

$$(ii) \nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla . \bullet \mathbf{B}) - \mathbf{B}(\nabla . \bullet \mathbf{A}) + (\mathbf{B} \bullet . \nabla)\mathbf{A} - (\mathbf{A} \bullet . \nabla)\mathbf{B}$$

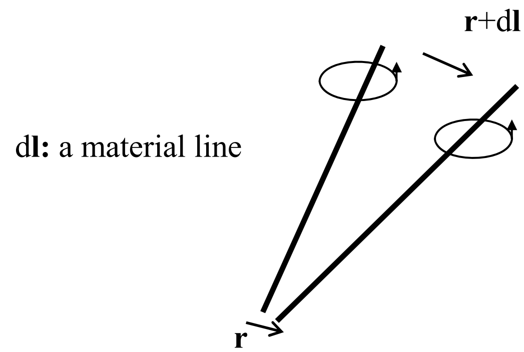
we finally have the vorticity equation

$$\partial_t \zeta_a + (\mathbf{u} \bullet \nabla)\zeta_a = -\zeta_a(\nabla \bullet . \mathbf{u}) + (\zeta_a . \bullet \nabla)\mathbf{u} + \frac{1}{\rho^2}\nabla\rho \times \nabla P + \nabla \times (\frac{1}{\rho}\mathbf{F}) \qquad (4.2.1)$$

local variability	adv.	stretching	distortion	Baroclinic term	Curl(forcing)

Compared with shallow water case, the two new terms are the distortion term and the baroclinic term.

The distortion term includes the two effects of rotating and stretching of the vortex tube.



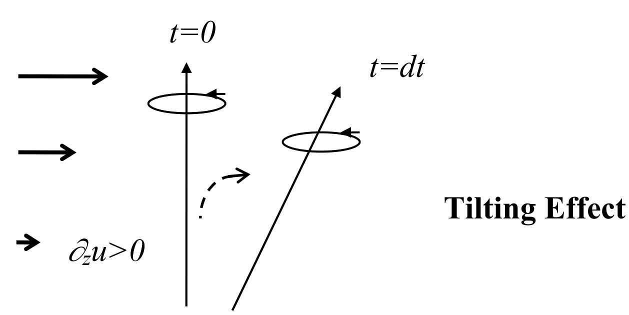
Following a vortex tube as a material line, we have

$$\frac{d}{dt}(d\mathbf{l}) = \frac{d}{dt}[(\mathbf{r}+d\mathbf{l})-\mathbf{r}] = \mathbf{u}(\mathbf{r}+d\mathbf{l})-\mathbf{u}(\mathbf{r}) = (d\mathbf{l}\bullet\nabla)\mathbf{u}$$

where $d\mathbf{l} \propto \boldsymbol{\zeta}_a$ is a material line following the vortex tube and defined in the local coordinate as the direction of z, such that $\boldsymbol{\zeta}_a = (0,0,\zeta_a)$. Therefore, the distortion term is

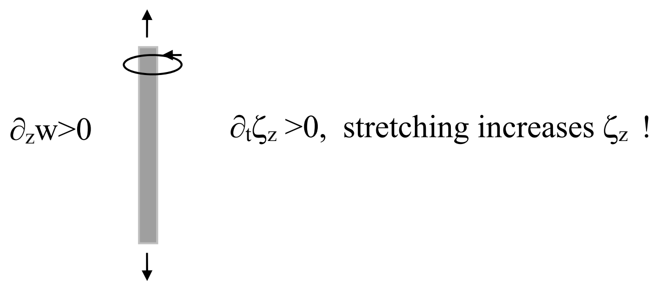
$$\frac{d}{dt}\boldsymbol{\zeta}_a \approx (\boldsymbol{\zeta}_a.\bullet\nabla)\mathbf{u} = (\zeta_a\partial_z)\mathbf{u} = \zeta_a\partial_z\mathbf{u} = (\underbrace{\zeta_a\partial u_z, \zeta_a\partial_z v}_{\text{tilting}}, \underbrace{\zeta_a\partial_z w}_{\text{stretching}}) \qquad (4.2.2)$$

The first two components represent the tilting effect, while the last component represents the stretching effect.



The tilting effect changes the direction of the vorticity. Consider a vortex tube, originally assumed in the z direction. The shear flow will tilts it down the direction of the shear, generating vorticity in the horizontal direction $\partial_t \zeta_x > 0$. This is like a helicopter. The helicopter moves forward when its propeller axis tilts horizontally. As such, the rotation of the propeller has a component in the horizontal direction.

The stretching effect has been seen in the shallow water system. When the vortex tube is stretched ($\partial_z w>0$) as in the figure above, the magnitude of the vorticity is increased ($\partial_t \zeta_z>0$). This is like the case of figure skating. The stretching effect does not change the direction of the vorticity.

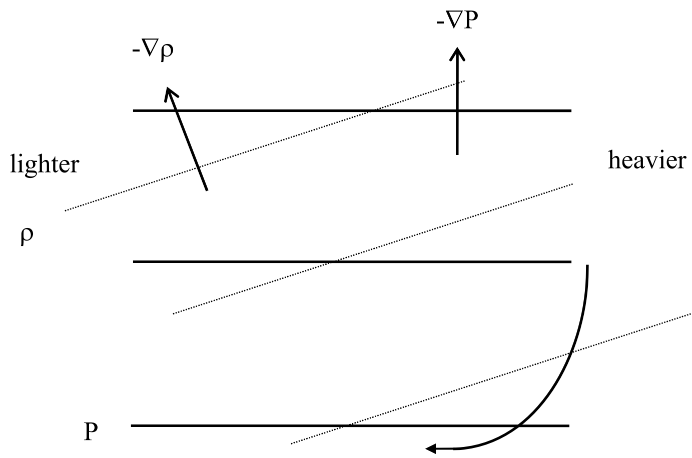


Note 1: The stretching of the distortion term is similar to the general stretching $\boldsymbol{\zeta}_\mathbf{a}(\nabla \bullet \mathbf{u})$. Indeed, take $\boldsymbol{\zeta}_\mathbf{a}=\zeta_a\mathbf{k}$, we have the sum of the distortion and general stretching terms as

$$\begin{aligned}\boldsymbol{\zeta}_a(\nabla \bullet \mathbf{u}) + (\boldsymbol{\zeta}_a \bullet \nabla)\mathbf{u} &= -\mathbf{k}\zeta_a(\partial_x u + \partial_y v + \partial_z w) + \mathbf{i}\zeta_a \partial_z u + \mathbf{j}\zeta_a \partial_z v + \mathbf{k}\zeta_a \partial_z w \\ &= [\zeta_a \partial_z u, \zeta_a \partial_z v, -\zeta_a(\partial_x u + \partial_y v)]\end{aligned}$$

The net effect is a "horizontal" (normal to ζ_a) convergence or divergence, which in the case of incompressible fluid $\partial_x u + \partial_y v = -\partial_z w$, is the same as the stretching effect. In fact, with incompressibility, the general stretching term $-\boldsymbol{\zeta}_a(\nabla \bullet \mathbf{u}) = 0$. The stretching effect comes only from the $\mathbf{z}$ component of the distortion term.

The effect of the baroclinic vorticity generation term can be seen in the following example. The differential density along the isobar creates horizontal density gradient. As such, the heavier fluid will sink and the lighter fluid rises, generating a clock wise rotation.



One important special case is the barotropic fluid,

$$p = p(\rho). \tag{4.2.3}$$

Now, the baroclinic term vanishes because $\nabla\rho \times \nabla p = \nabla\rho \times \frac{dp}{dp}\nabla\rho \equiv 0$.

In fact, (4.2.3) is the original definition of barotropic fluid, while ρ=const is the special case of the barotropic fluid.

2: Circulation Theorem

In Section 1.3, we have seen that the total circulation is conserved in the homogeneous fluid if forcing and dissipation are neglected. In the baroclinic case, the circulation is no longer conserved.

Follow the derivation in Section. 1.3, but keep the term $\frac{1}{\rho}\nabla p$, we have

$$\frac{d}{dt}\oint \mathbf{u} \bullet d\mathbf{l} = \oint \frac{d\mathbf{u}}{dt} \bullet d\mathbf{l} = \oint 2\Omega \times \mathbf{u} \bullet d\mathbf{l} - \oint \frac{1}{\rho}\nabla p \bullet d\mathbf{l} + \oint \frac{\mathbf{F}}{\rho} \bullet d\mathbf{l}$$

Since

$$\oint 2\Omega \times \mathbf{u} \bullet d\mathbf{l} = 2\Omega \frac{dA_n}{dt},$$

$$\oint \frac{1}{\rho}\nabla p \bullet d\mathbf{l} = \iint \nabla \times (\frac{\nabla p}{\rho}) \bullet \vec{n} dA = -\iint \frac{\nabla\rho \times \nabla p}{\rho^2} \bullet \vec{n} dA$$

We have the circulation equation

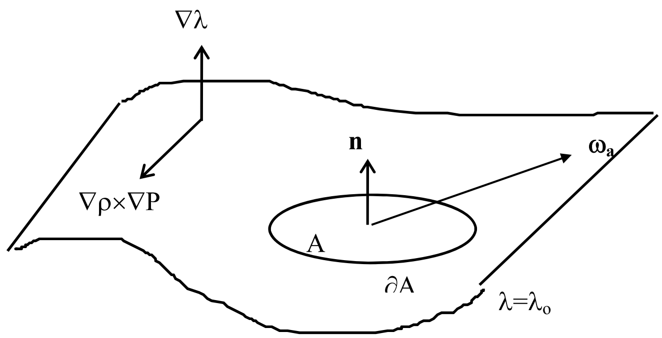
$$\frac{d}{dt}\Gamma_a = \frac{d}{dt}\left(\Gamma + 2\Omega A_n\right) = \iint \frac{\nabla\rho \times \nabla p}{\rho^2} \bullet \vec{n} dA + \oint \frac{\mathbf{F}}{\rho} \bullet d\mathbf{l} \qquad (4.2.4)$$

The Kelvin's Theorem can then be modified: the total circulation is conserved if the baroclinic term vanishes, or equivalently, the fluid has to be barotropic $p = p(\rho)$. Similarly, there is no Bernoulli equation in the presence of baroclinic term.

Section 4.3: Ertel Potential Vorticity

1 Ertel Potential Vorticity Conservation

Potential vorticity has been seen of critical importance in the shallow water dynamics. The concept of the shallow water potential vorticity (Rossby 1940) can be generalized to a stratified fluid (Ertel, 1942). Here, we will study the Ertel PV in its integral form, making use of the Kelvin's theorem. A more detailed derivation can be found in Pedlosky (Ch 2).



At a first sight, the Kelvin's theorem (4.2.4) states that the total circulation is not conserved in the presence of baroclinity. This is a serious limitation on GFD applications because all our fluids are stratified and therefore strongly baroclinic. However, an alternate quantity, called the Ertel potential vorticity can still be derived from the Kelvin's theorem. As in the shallow water case, this quantity is of fundamental importance to GFD!

Neglecting forcing and dissipation, and using the Stoke's theorem, the Kelvin's theorem can be written as

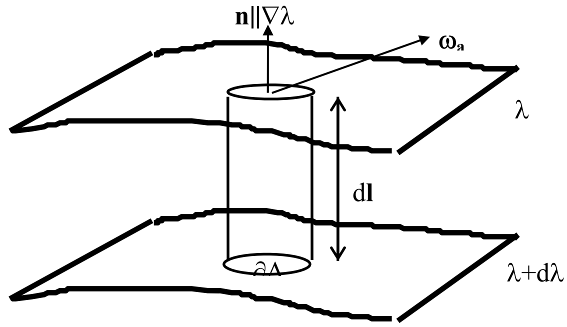
$$\frac{d}{dt}\iint_A \zeta_a \bullet \mathbf{n} dA = \iint_A \left(\frac{\nabla \rho \times \nabla p}{\rho^2} \right) \bullet \mathbf{n} dA \qquad (4.3.1)$$

It is seen that the baroclinic term that prohibits the conservation of the circulation. However, as seen below, by carefully selecting our integral surface, the baroclinic term can be eliminated. We choose a quantity λ which is a material surface ($\frac{d\lambda}{dt} = 0$), and consider the circulation on a surface A that is on a constant λ surface. Furthermore, we assume $\lambda = \lambda(\rho, p)$, such that **n** is parallel to $\nabla\lambda = \frac{\partial\lambda}{\partial\rho}\nabla\rho + \frac{\partial\lambda}{\partial p}\nabla p$. This leads to:

$$n \bullet (\nabla\rho \times \nabla p) = \nabla\lambda \bullet (\nabla\rho \times \nabla p) = 0$$

Therefore, (4.3.1) becomes

$$\frac{d}{dt}\iint_{A(\lambda=const)} \zeta_a \bullet \mathbf{n} dA = 0$$



When the area of the surface element $A \to 0$, we have approximately

$$\frac{d}{dt}(\zeta_a \bullet \mathbf{n}\delta A) = 0 \qquad (4.3.4)$$

Since λ is a material surface, mass conservation states that

$$\Delta m = \rho\delta l\delta A = \rho\frac{\delta\lambda}{|\nabla\lambda|}\delta A = \text{const.} \qquad (4.3.5)$$

following the flow. (we have used $\delta\lambda = |\nabla\lambda|\delta l$, *whose* one-dimension analogy is $\delta\lambda = \partial_x\lambda\delta x$). Also, we can write the normal vector as

$$\mathbf{n} = \frac{\nabla\lambda}{|\nabla\lambda|}. \qquad (4.3.6)$$

Substitute δA and **n** from (4.3.5) and (4.3.6) in (4.3.4), we have the Ertel potential voriticity conservation

$$\frac{d\Pi}{dt} = 0, \qquad (4.3.7)$$

where

$$\Pi = \zeta_a \bullet \frac{\nabla\lambda}{\rho} \qquad (4.3.7a)$$

is the Ertel potential vorticity. The derivation above shows that the conservation of Ertel PV is a direct result of the Kelvin's theorem, but on a special material surface $\lambda=\lambda(\rho,p)$. In other words, the conditions for the conservation of Ertel PV are:

$$\text{i)} \begin{cases} \frac{d\lambda}{dt} = 0 \text{ (material surface)} \\ \lambda = \lambda(\rho, p), \quad \text{e.g. } \lambda = \rho \end{cases}$$

ii) no forcing or dissipation to the system.

Here are some most commonly used Ertel PVs. In the ocean, since $d\rho/dt=0$ in the adiabatic flow, we set $\lambda = \frac{\rho^2}{2}$, and the Ertel PV becomes

$$\Pi = \zeta_a \bullet \nabla\rho \approx \zeta_a \frac{\partial\rho}{\partial z},$$

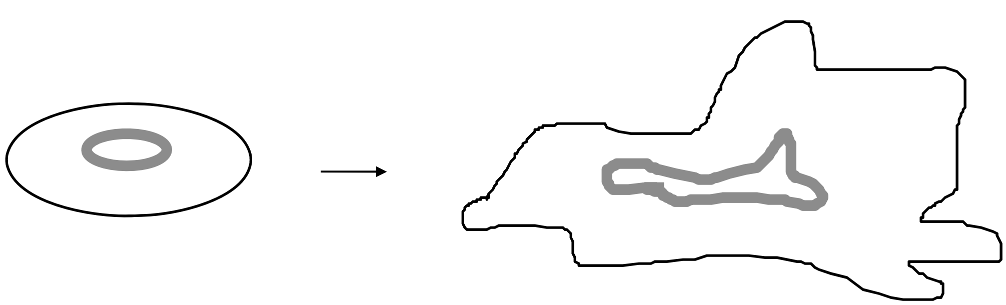
where we have used $\rho_z >> \rho_x, \rho_y$. If furthermore, $\zeta_a \approx f$ because $\zeta/f=\varepsilon<<1$, we have the planetary potential voriticity

$$\Pi \approx f \frac{\partial\rho}{\partial z}.$$

In the atmosphere, $\frac{d\theta}{dt} = 0$ in the adiabatic flow, we set $\lambda = -\frac{\theta}{g}$ and the Ertel PV becomes

$$\Pi = -\frac{\zeta_a \bullet \nabla\theta}{g\rho} \approx -\frac{\zeta_a \partial_z \theta}{g\rho} = \zeta_a \frac{\partial\theta}{\partial p}$$

The conservation of PV provides a powerful constrain on the course of the motion of a fluid parcel. In the mean time, PV can also be used for diagnostic studies in the observation. Finally, it can also be used as a tracer to track the long term particle motion.



Questions for Chapter 4

Q4.1: With hydrostatic approximation, prove mathematically that the pressure gradient force in the Z_g coordinate is transferred to the geopotential height in the p (or $Z \approx - \ln p$ coordinate).

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_{Z_g} = \left(\frac{\partial \Phi}{\partial x}\right)_Z$$

here $p=p(x, y, Z_g, t)$ is the pressure in the Z_g coordinate and $\Phi = \Phi(x,y,Z,t)$ is the geopotential height in the p *(or Z)* coordinate.

Q4.2: The stretching term and tilting term are important in the vorticity equation (4.2.1), but play no role in the circulation theory (4.2.4). Why?

Exercises for Chapter 4

E4.1: (f-plane waves) For the continuously stratified fluid satisfying the general equations (4.15), small perturbation satisfies the linearized equations

$$\partial_t u - fv = -\frac{\partial_x p}{\rho_m}$$
$$\partial_t v + fu = -\frac{\partial_y p}{\rho_m}$$
$$\partial_t w + \frac{g\rho}{\rho_m} = -\frac{\partial_z p}{\rho_m}$$
$$\partial_x u + \partial_y v + \partial_z w = 0$$
$$\partial_t \rho + w\overline{\rho}_z = 0$$

on a f-plane, for a uniform stratification (such that $N^2 = -\frac{g\overline{\rho}_z}{\rho_m}$ is a positive constant)

(a) Show that the dispersion relationship for the waves of the form $\exp[i(kx + ly + mz + \omega t)]$are:

$$\omega_1 = 0$$

for the geostrophic (vorticity) mode, and

$$\omega_{2,3}^2 = \frac{N^2(k^2 + l^2) + f^2 m^2}{k^2 + l^2 + m^2}$$

for the internal inertial-gravity waves.

(b) Under hydrostatic approximation such that the vertical momentum equation is $\frac{g\rho}{\rho_m} = -\frac{\partial_z p}{\rho_m}$, show that the dispersion relationships are

$$\omega_1 = 0,$$

for the geostrophic (vorticity) mode, and

$$\omega_{2,3}^2 = \frac{N^2(k^2 + l^2) + f^2 m^2}{m^2}$$

for the internal internal-gravity waves. What is the relation between the dispersion relationships of this internal inertial-gravity wave and the inertial-gravity wave derived from the 1.5-layer model?

(c) Under what conditions, you think that the hydrostatic approximation is not good because it distorts the wave dispersion relationships too much.

(d) In a homogeneous fluid, what are the dispersion relationships for these waves with and without hydrostatic approximation?

E4.2: (PV of -f-plane waves) The linarized Ertel PV can be derived from the equations in ***E4.1*** as $\Pi = [(f+\zeta)\frac{\partial \rho}{\partial z}]' = \zeta' \overline{\rho}_z + f\frac{\partial \rho'}{\partial z}$, Using the equations and results in ***E4.1***, derive the PV disturbances corresponding to the geostrophic vorticity mode and the internal inertial-gravity wave mode.

(a) For the geostrophic mode, show that we can define a streamfunction ψ, such that

$$\Pi = \overline{\rho}_z \mathrm{q}_{\mathrm{QGPV}}, \qquad \text{and} \qquad q_{QGPV} = \nabla^2{}_H \psi + \partial_z \left(\frac{f_o^2}{N^2}\frac{\partial \psi}{\partial z}\right)$$

where $\mathrm{q}_{\mathrm{QGPV}}$ is the perturbation QGPV, which can be (see Chapter 5) derived from eqn. (5.1.20b).

(b) For inertial-gravity wave modes, show that the PV is zero (regardless of hydrostatic approximation)

$$\Pi = 0$$

(hint: With the wave form $\exp[i(kx+ly+mz+\omega t)]$ in ***E4.1***, using the eigenfunctions derived from the four equations

$$\partial_t u - fv = -\frac{\partial_x p}{\rho_m}, \quad \partial_t v + fu = -\frac{\partial_y p}{\rho_m}, \quad \partial_x u + \partial_y v + \partial_z w = 0, \quad \partial_t \rho + w\overline{\rho}_z = 0$$

such that u and v can be represented in terms of density)

E4.3: For the continuously stratified fluid satisfying the equations (4.15), the low frequency, large scale, small perturbation satisfies the linearized equations

$$-fv = -\frac{1}{\rho_0}\partial_x p + \frac{1}{\rho_0}F_x$$
$$+fu = \frac{1}{\rho_0}\partial_y p + \frac{1}{\rho_0}F_y$$
$$\partial_z p = -g\rho$$
$$\partial_x u + \partial_y v + \partial_z w = 0$$
$$\partial_t \rho + w\overline{\rho}_z = S_0$$

where we have dropped ' for the perturbation, and $\overline{\rho}_z$ is the mean stratification.
(a) Show that the equations can be reduced to a single equation for the perturbation density as

$$\partial_t\left(\frac{\rho}{\overline{\rho}_z}\right)_{zz} - \frac{\beta g}{f^2\rho_0}\rho_x = curl\left(\frac{\vec{F}_z}{\rho_0 f}\right) + \left(\frac{S_0}{\overline{\rho}_z}\right)_{zz}$$

(b) Assume $\overline{\rho}_z = const$, and the momentum and density forcing vanish, the free mode satisfies

$$\partial_{tzz}\rho + \frac{\beta N^2}{f^2}\rho_x = 0$$

where $N^2 = -\frac{g}{\rho_0}\overline{\rho}_z$ represents the Brunt-Vasara frequency. Assume the perturbation has the form of a plane wave $\rho \propto \exp[i(mz + kx - \omega t)]$, derive the dispersion relationship of the wave. Can you guess what wave is this?

<u>**E4.4**: (Monsoon-Desert, or Stationary atmospheric response to deep heating in the subtropics)</u>: On a subtropical beta-plane, $f = f_0 + \beta y$, with a mean vertical potential temperature profile $\Theta(z)$, a large scale atmospheric circulation is forced by a steady diabatic heating *Q*. With proper nondimensionalization, the stationary atmospheric response satisfies the following set of equations

$$\begin{cases} -fV = -\partial_x P & (1a) \\ +fU = -\partial_y P & (1b) \\ \partial_z P = T & (1c) \\ U_x + V_y + W_z = 0 & (1d) \\ WN^2 = -\varepsilon\theta + Q & (1e) \end{cases}$$

where $N^2 = \frac{g\partial_z\Theta}{\Theta}$ is the Brunt-Vasala frequency, and *(U, V), W, P, W, T*are winds, vertical velocity, pressure and potential temperature, respectively. The steady state is achieved by a thermal damping ε, which is associated with the long wave cooling.

1)For a deep heating in the middle atmosphere, associated with an latent heating of a summer monsoon, what do you expect the vertical structure of the forced atmospheric response is? barotropic, equivalent barotropic or baroclinic?
2)For the baroclinic response, a student assumed that the response of the atmosphere can be approximated in the vertical in 3 levels as shown below

Level 3 -------------------------- $-u, -v, -p, w = 0$ upper level (*z=2h*, ~200mb)

Level 2 -------------------------- $q, \theta, w,$ mid-level ($z=h$, ~500mb)

Level 1 -------------------------- $u, v, p, w=0$ lower-level ($z=0$, ~850mb)

Fig.1: The three-level model

He then approximate equations (1a-e) using the finite-difference in the vertical. In terms of the lower level winds (u,v), pressure (p), and middle level diabatic heating q, potential temperature θ and vertical velocity w, please show that the baroclinic response is now determined by the following equations

$$\begin{cases} -fv = -\partial_x p & (2a) \\ +fu = -\partial_y p & (2b) \\ p = -h\theta & (2c) \\ u_x + v_y + w/h = 0 & (2d) \\ wN^2 = -\varepsilon\theta + q & (2e) \end{cases}$$

3)For a localized heating in x,

$$q(x) = \begin{cases} q_0 > 0, & 0 \le x \le L \\ 0, & elsewhere \end{cases} \qquad (3)$$

Find the forced atmospheric circulation. Discuss the spatial structure of the response, in terms of the pressure, wind, temperature, and vertical velocity.

4)Do you expect a desert region to be generated by this monsoon heating? If you do, where is it (relative to the heating) and why?

5)Give examples where you think this monsoon-desert mechanism applies to our real world.

(for mathematic convenience, you can assume $N=1,\ h=1$). (plot the solution for $\varepsilon=1$, $c = \beta N^2 h^2 / f^2 = 1$).